Fundamental groups, Alexander invariants, and cohomology jumping loci

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Dedicated to Anatoly Libgober on the occasion of his sixtieth birthday

ABSTRACT. We survey the cohomology jumping loci and the Alexander-type invariants associated to a space, or to its fundamental group. Though most of the material is expository, we provide new examples and applications, which in turn raise several questions and conjectures.

The jump loci of a space X come in two basic flavors: the characteristic varieties, or, the support loci for homology with coefficients in rank 1 local systems, and the resonance varieties, or, the support loci for the homology of the cochain complexes arising from multiplication by degree 1 classes in the cohomology ring of X. The geometry of these varieties is intimately related to the formality, (quasi-) projectivity, and homological finiteness properties of $\pi_1(X)$.

We illustrate this approach with various applications to the study of hyperplane arrangements, Milnor fibrations, 3-manifolds, and right-angled Artin groups.

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1. Introduction

1.1. Fundamental groups. The fundamental group of a topological space was introduced in 1904 by H. Poincaré, with the express purpose of distinguishing between certain manifolds, such as the dodecahedral space and the three-sphere, which otherwise share a lot in common. Subsequently, it was realized that the geometric nature of a manifold M influences the group-theoretic properties of its fundamental group, $\pi_1(M)$; and conversely, the nature of a group G can say a lot about those manifolds with fundamental group G, at least in low dimensions.

As is well-known, every finitely presented group G occurs as the fundamental group of a smooth, compact connected, orientable manifold M of dimension $n = 4$. The manifold can even be chosen to be symplectic (Gompf); and, if one is willing to go to dimension $n = 6$, it can be chosen to be complex (Taubes). On the other hand, requiring M to be 3-dimensional, or to carry a Kähler structure puts severe restrictions on its fundamental group.

A finitely presented group G is said to be a *Kähler group* if it can be realized as $G = \pi_1(M)$, where M is a compact, connected Kähler manifold. The notions of quasi-Kähler group and (quasi-) projective group are defined similarly. A classical problem, formulated by J.-P. Serre in the late 1950s (cf. [80]), is to determine which groups can be so realized.

A basic theme of this survey is that certain topological invariants, such as the Alexander polynomial, or the cohomology jumping loci, are very good tools for studying questions about fundamental groups in algebraic geometry. In particular, they can be brought to bear to settle Serre's realization problem for several notable classes of groups.

1.2. Alexander-type invariants. In his foundational paper [1], J. W. Alexander assigned to every knot K in S^3 a polynomial with integer coefficients, as follows. Let $\pi \colon X' \to X$ be the infinite cyclic cover of the knot complement. Then $A = H_1(X', \pi^{-1}(x_0), \mathbb{Z})$ is a finitely presented module over the group ring $\mathbb{Z}\mathbb{Z} \cong \mathbb{Z}[t^{\pm 1}]$. The Alexander polynomial, $\Delta_K(t)$, equals—up to normalization—the greatest common divisor of the codimension 1 minors of a presentation matrix for the Alexander module A. Clearly, $\Delta_K(t)$ depends only on the knot group, $G = \pi_1(X, x_0)$. More generally, if K is an n-component link in S^3, there is a multi-variable Alexander polynomial $\Delta_K(t_1, \ldots, t_n)$, depending only on the link group, and a choice of meridians.

These definitions extend with only slight modifications to arbitrary finitely presented groups (see §4). In [54], A. Libgober considered the single-variable Alexander polynomial $\Delta_C(t)$ associated to the complement of an algebraic curve $C \subset \mathbb{C}^2$, and its total linking cover. He then proved in [56] that all the roots of $\Delta_C(t)$ are roots of unity, a result which constrains the class of groups realizable as fundamental groups of plane curve complements.

As shown in [**29**, **30**], the multi-variable Alexander polynomial and the related Alexander varieties provide further constraints on the kind of fundamental groups that can appear in this algebraic context. For example, if G is a quasi-projective group with $b_1(G) \neq 2$, then the Newton polytope of Δ_G is a line segment; and, if G is a projective group, then Δ_G must be constant (see Theorem 8.10).

1.3. Cohomology jumping loci. An essential tool for us are the cohomology jumping loci associated to a connected, finite-type CW-complex X. The *characteristic varieties* $\mathcal{V}_d^i(X, \Bbbk)$ are algebraic subvarieties of $\mathrm{Hom}(\pi_1(X), \Bbbk^{\times})$, while the *resonance varieties* $\mathcal{R}_d^i(X, \Bbbk)$ are homogeneous subvarieties of $H^1(X, \Bbbk)$. The former are the jump loci for the homology of X with rank 1 twisted complex coefficients (see §2), while the latter are the jump loci for the homology of the cochain complexes arising from multiplication by degree 1 classes in $H^*(X, \Bbbk)$ (see §5). The jump loci of a group are defined in terms of the jump loci of the corresponding classifying space.

It has been known since the work of Hironaka [**51**] that the degree 1 characteristic varieties of a finitely generated group G coincide with the determinantal varieties of its Alexander matrix, at least away from the origin. A more general statement, valid in arbitrary degrees, was recently proved in [**74**] (see Theorem 4.6).

Foundational results on the structure of the characteristic varieties of Kähler and quasi-Kähler manifolds and smooth projective and quasi-projective varieties were obtained by Beauville [**6**], Green–Lazarsfeld [**49**], Simpson [**82**], Campana [**11**], and ultimately, Arapura [**3**]. In particular, if $X = \overline{X} \setminus D$, with $\overline{X}$ compact Kähler, $b_1(\overline{X}) = 0$, and D a normal-crossings divisor, then each variety $\mathcal{V}_d^i(X, \mathbb{C})$ is a union of subtori of $\mathrm{Hom}(\pi_1(X), \mathbb{C}^{\times})$, possibly translated by unitary characters (see §8). In [**57**], Libgober—who coined the name "characteristic varieties"—showed how to find irreducible components of $\mathcal{V}_d^1(X, \mathbb{C})$ from the faces of a certain "polytope of quasiadjunction," in the case when X is a plane curve complement. Recently, he proved in [**59**] a local version of Arapura's theorem, in which all the translations are done by characters of finite order.

The resonance varieties were first defined by Falk in [**39**], in the case when $X = X(\mathcal{A})$ is the complement of a complex hyperplane arrangement $\mathcal{A}$. Best understood are the degree 1 resonance varieties $\mathcal{R}_d(\mathcal{A}) = \mathcal{R}_d^1(X(\mathcal{A}), \mathbb{C})$; these varieties admit a very precise combinatorial description, owing to work of Falk [**39**], Cohen–Suciu [**18**], Libgober [**57**], Libgober–Yuzvinsky [**60**], and others, with the state of the art being the work of Falk, Pereira, and Yuzvinsky [**41**, **75**, **91**].

The cohomology jump loci provide a unifying framework for the study of a host of questions, both quantitative and qualitative, concerning a space X and its fundamental group, $G = \pi_1(X)$. A lot of the recent developments described in this survey come from our joint work with A. Dimca and S. Papadima [**29**, **30**, **31**, **32**, **72**, **74**].

1.4. Abelian covers. As shown by Libgober in [**55**], with further refinements by Hironaka [**50**], Sarnak–Adams [**79**], Sakuma [**78**], and Matei–Suciu [**65**], counting torsion points on the character group, according to their depth with respect to the stratification by the characteristic varieties, yields very precise information about the homology of finite abelian covers of X (see §3).

An old observation in knot theory is that the Alexander polynomial $\Delta_K(t)$ of a fibered knot K is monic. More generally, Dwyer and Fried [**34**] showed that

the support varieties of the Alexander invariants of a finite CW-complex completely determine the homological finiteness properties of its free abelian covers. This result was recast in [**74**] in terms of the characteristic varieties and their (exponential) tangent cones (see Theorem 3.2).

In [**8**], Bieri, Neumann, and Strebel associated to every finitely generated group G an open, conical subset $\Sigma^1(G)$ of the real vector space $\operatorname{Hom}(G, \mathbb{R})$. The BNS invariant and its higher-order generalizations, the invariants $\Sigma^q(G, \mathbb{Z})$ of Bieri and Renz [**9**], hold subtle information about the homological finiteness properties of normal subgroups of G with abelian quotients (see §6). The actual computation of the Σ-invariants is enormously complicated. Yet, as shown in [**74**], the cohomology jumping loci of a classifying space $K(G, 1)$ provide computable upper bounds for the Σ-invariants of G (see Theorems 6.1 and 6.2).

1.5. Formality and the tangent cone formula. A central point in the theory of cohomology jumping loci is the relationship between the characteristic and resonance varieties. As proved by Libgober in [**58**], the tangent cone to $\mathcal{V}_d^i(X, \mathbb{C})$ at the origin is contained in $\mathcal{R}_d^i(X, \mathbb{C})$, for any finite-type CW-complex X. But, as first noted in [**65**], the inclusion can be strict; in fact, as shown in [**30**], equality may even fail for quasi-projective groups.

The crucial property that bridges the gap between the tangent cone to a characteristic variety and the corresponding resonance variety is *formality* (see §7). A space X as above is formal if its Sullivan minimal model is quasi-isomorphic to $(H^*(X, \mathbb{Q}), 0)$, while a finitely generated group G is 1-formal if its Malcev Lie algebra has a quadratic presentation. For a recent survey of these notions, we refer to [**73**].

One of the main results from [**30**] establishes an isomorphism between the analytic germ of $\mathcal{V}_d^1(G, \mathbb{C})$ at 1 and the analytic germ of $\mathcal{R}_d^1(G, \mathbb{C})$ at 0, in the case when G is a 1-formal group (see Theorem 7.1). This isomorphism provides a new, and very effective 1-formality criterion.

As also shown in [**30**], the tangent cone theorem, when used in conjunction with Arapura's theorem, imposes very strong restrictions on the nature of the resonance varieties of Kähler groups, and, more generally, 1-formal, quasi-Kähler groups (see Theorem 8.5). These restrictions lead to both quasi-projectivity obstructions (in the formal setting), and enhanced formality obstructions (in the quasi-projective setting).

1.6. Applications. The techniques described above have a large range of applicability. We illustrate their usefulness with a variety of examples, arising in algebraic geometry, low-dimensional topology, combinatorics, and group theory.

As a running example, we use an especially well-suited class of combinatorially defined spaces. A simplicial complex L on n vertices determines a subcomplex T_L of the n-torus, with fundamental group the right-angled Artin group G_Γ corresponding to the graph $\Gamma = L^{(1)}$. The cohomology jumping loci of these "toric complexes" were computed in [**72**]. Using the computation of $\mathcal{R}_1^1(G_\Gamma, \mathbb{C})$, it was shown in [**30**] that a group G_Γ is quasi-projective if and only if Γ is a complete multi-partite graph. We show here that the Alexander polynomial of G_Γ is non-constant if and only if Γ has connectivity 1 (see Proposition 4.13).

In §9, we treat in detailed fashion an important class of quasi-projective varieties: those arising as complements of complex hyperplane arrangements. The

cohomology jumping loci of such a complement, $X = X(\mathcal{A})$, are rather well understood, especially in degree 1, with a major open problem being the determination of the translated components in $\mathcal{V}_1^1(X(\mathcal{A}), \mathbb{C})$. Much effort has been put over the years into comprehending fundamental groups of arrangements (see [83] for more on this subject). We identify here precisely the class of line arrangements $\mathcal{A}$ in $\mathbb{C}^2$ for which the group $G(\mathcal{A}) = \pi_1(X(\mathcal{A}))$ is a Kähler group (see Theorem 9.11), respectively, a right-angled Artin group (see Theorem 9.12). Along the way, we reprove a recent result of Fan [43], identifying those line arrangements $\mathcal{A}$ for which $G(\mathcal{A})$ is a free group (see Theorem 9.10).

In §10, we discuss Milnor fibrations, with special emphasis on those arising from hyperplane arrangements. A well-known construction, due to J. Milnor [68], associates to a weighted homogeneous polynomial $f \in \mathbb{C}[z_0, \ldots, z_d]$ of degree n a smooth fibration, $f\colon X \to \mathbb{C}^{\times}$, where X is the complement in $\mathbb{C}^{d+1}$ of the hypersurface $\{f = 0\}$. The Milnor fiber, $F = f^{-1}(1)$, is an n-fold cyclic cover of $U = X/\mathbb{C}^{\times}$. We review this construction, and the method of computing the homology groups $H_1(F, \mathbb{k})$ from the characteristic varieties $\mathcal{V}_d^1(U, \mathbb{k})$, provided $(\operatorname{char} \mathbb{k}, n) = 1$. We also review recent progress on the formality question for the Milnor fiber, first raised in [73], and recently answered—with different examples—by Zuber [92] and Fernández de Bobadilla [46].

We conclude in §11 with a look at the world of 3-dimensional manifolds. A major application of the techniques discussed in this survey is the solution given in [32] to the following question asked by S. Donaldson and W. Goldman in 1989, and independently by A. Reznikov in 1993: Which 3-manifold groups are Kähler groups? The proof uses in an essential way the contrasting nature of the resonance varieties for these two classes of groups. Another application, given in [31], is to the classification (up to Malcev completion) of 3-manifold groups which are also 1-formal, quasi-Kähler groups. This classification can be made very precise in the case of boundary manifolds of line arrangements in $\mathbb{CP}^2$: as shown in [20] and [29], the only way the fundamental group of a boundary manifold $M(\mathcal{A})$ can be either 1-formal, or quasi-Kähler (or both), is for $\mathcal{A}$ to be a pencil, or a near-pencil.

2. Characteristic varieties

We start with the jumping loci for homology in rank 1 local systems, and two types of tangent cones associated to a subvariety of the character variety of a group.

2.1. Rank 1 local systems. Let X be a connected CW-complex, with finite k-skeleton, for some $k \geq 1$. Without loss of generality, we may assume X has a single 0-cell, call it x_0. Moreover, we may assume all attaching maps $(S^i, *) \to (X^i, x_0)$ are basepoint-preserving.

Fix a field $\mathbb{k}$, and denote by $(C_i(X, \mathbb{k}), \partial_i)_{i \geq 0}$ the cellular chain complex of X. Let $p\colon \widetilde{X} \to X$ be the universal cover. The cell structure on X lifts in a natural fashion to a cell structure on $\widetilde{X}$. Fixing a lift $\tilde{x}_0 \in p^{-1}(x_0)$ identifies the fundamental group, $G = \pi_1(X, x_0)$, with the group of deck transformations of $\widetilde{X}$, which permute the cells. Therefore, we may view $(C_i(\widetilde{X}, \mathbb{k}), \tilde{\partial}_i)_{i \geq 0}$ as a chain complex of left-modules over the group ring $\mathbb{k}G$.

Let $\mathbb{k}^{\times}$ be the group of units in $\mathbb{k}$. The group of $\mathbb{k}$-valued characters, $\operatorname{Hom}(G, \mathbb{k}^{\times})$, is an algebraic group, with pointwise multiplication inherited from $\mathbb{k}^{\times}$, and identity the character taking constant value $1 \in \mathbb{k}^{\times}$ for all $g \in G$. This character group

parametrizes rank 1 local systems on X: given a character $\rho\colon G \to \Bbbk^\times$, denote by $\Bbbk_\rho$ the 1-dimensional $\Bbbk$-vector space, viewed as a right module over the group ring $\Bbbk G$ via $a \cdot g = \rho(g)a$, for $g \in G$ and $a \in \Bbbk$. The homology groups of X with coefficients in $\Bbbk_\rho$ are defined as

$$(1) \qquad H_i(X, \Bbbk_\rho) := H_i(C_\bullet(\widetilde{X}, \Bbbk) \otimes_{\Bbbk G} \Bbbk_\rho).$$

In this setup, the identity $1 \in \operatorname{Hom}(G, \Bbbk^\times)$ corresponds to the trivial coefficient system, $\Bbbk_1 := \Bbbk$, and $H_*(X, \Bbbk)$ is the usual homology of X with $\Bbbk$-coefficients.

2.2. Homology jump loci. Computing homology groups with coefficients in rank 1 local systems leads to a natural filtration of the character group.

Definition 2.1. The *characteristic varieties* of X (over $\Bbbk$) are the Zariski closed sets

$$\mathcal{V}_d^i(X, \Bbbk) = \{\rho \in \operatorname{Hom}(G, \Bbbk^\times) \mid \dim_\Bbbk H_i(X, \Bbbk_\rho) \geq d\},$$

defined for all integers $0 \leq i \leq k$ and $d > 0$.

When computing the characteristic varieties in degrees up to k, we may assume, without loss of generality, that X is a finite CW-complex of dimension $k+1$; see [74, Lemma 2.1] for an explanation. In each fixed degree i, the characteristic varieties define a descending filtration on the character group,

$$(2) \qquad \operatorname{Hom}(G, \Bbbk^\times) \supseteq \mathcal{V}_1^i(X, \Bbbk^\times) \supseteq \mathcal{V}_2^i(X, \Bbbk^\times) \supseteq \cdots.$$

Clearly, $1 \in \mathcal{V}_1^i(X, \Bbbk)$ if and only if $H_i(X, \Bbbk) \neq 0$. In degree 0, we have $\mathcal{V}_1^0(X, \Bbbk) = \{1\}$ and $\mathcal{V}_d^0(X, \Bbbk) = \emptyset$, for $d > 1$. In degree 1, the characteristic varieties $\mathcal{V}_d^1(X, \Bbbk)$ depend only on the fundamental group $G = \pi_1(X)$—in fact, only on its maximal metabelian quotient, G/G''—so we sometimes denote them as $\mathcal{V}_d(G, \Bbbk)$.

Define the *depth* of a character $\rho\colon \pi_1(X, x_0) \to \Bbbk^\times$ relative to the stratification (2) by

$$(3) \qquad \operatorname{depth}_\Bbbk^i(\rho) = \max\{d \mid \rho \in \mathcal{V}_d^i(X, \Bbbk)\}.$$

As above, we abbreviate $\operatorname{depth}_\Bbbk(\rho) = \operatorname{depth}_\Bbbk^1(\rho)$.

In general, the characteristic varieties depend on the field of definition $\Bbbk$. Nevertheless, if $\Bbbk \subseteq \mathbb{K}$ is a field extension, then

$$(4) \qquad \mathcal{V}_d^i(X, \Bbbk) = \mathcal{V}_d^i(X, \mathbb{K}) \cap \operatorname{Hom}(G, \Bbbk^\times).$$

Thus, for practical purposes it is usually assumed that $\Bbbk$ is algebraically closed. We will often suppress the coefficient field in the default situation when $\Bbbk = \mathbb{C}$; for instance, we will write $\mathcal{V}_d^i(X) = \mathcal{V}_d^i(X, \mathbb{C})$.

The depth-1 characteristic varieties satisfy a simple product formula:

$$(5) \qquad \mathcal{V}_1^i(X_1 \times X_2, \Bbbk) = \bigcup_{p+q=i} \mathcal{V}_1^p(X_1, \Bbbk) \times \mathcal{V}_1^q(X_2, \Bbbk),$$

provided both X_1 and X_2 have finitely many i-cells, see [74, Proposition 13.1].

2.3. Toric complexes. Before developing the theory further, let us pause for some examples, showing how the characteristic varieties can be computed explicitly in favorable situations.

Example 2.2. Let S^1 be the unit circle in $\mathbb{R}^2$, with basepoint $* = (1, 0)$. Identify the fundamental group $\pi_1(S^1, x_0)$ with $\mathbb{Z}$, and its character group, $\text{Hom}(\mathbb{Z}, \mathbb{k})$, with $\mathbb{k}^\times$. It is readily seen that $\mathcal{V}_1^0(S^1, \mathbb{k}) = \mathcal{V}_1^1(S^1, \mathbb{k}) = \{1\}$, and $\mathcal{V}_d^i(S^1, \mathbb{k}) = \emptyset$, otherwise.

More generally, let $T^n = S^1 \times \cdots \times S^1$ be the n-torus. Upon identifying $\pi_1(T^n) = \mathbb{Z}^n$ and $\text{Hom}(\mathbb{Z}^n, \mathbb{k}) = (\mathbb{k}^\times)^n$, we find that $\mathcal{V}_d^i(T^n, \mathbb{k}) = \{1\}$, if $d \leq \binom{n}{i}$, and $\mathcal{V}_d^i(T^n, \mathbb{k}) = \emptyset$, otherwise.

Example 2.3. Now let $(T^n)^{(1)} = \bigvee^n S^1$ be the 1-skeleton of the n-torus, identified with the wedge of n copies of S^1 at the basepoint. Clearly, $\pi_1(\bigvee^n S^1) = F_n$, the free group of rank n, and $\text{Hom}(F_n, \mathbb{k}) = (\mathbb{k}^\times)^n$. It is readily seen that $\mathcal{V}_d^1(\bigvee^n S^1, \mathbb{k}) = (\mathbb{k}^\times)^n$ for $d \leq n - 1$, while $\mathcal{V}_n^1(\bigvee^n S^1, \mathbb{k}) = \{1\}$ and $\mathcal{V}_d^1(\bigvee^n S^1, \mathbb{k}) = \emptyset$ for $d > n$.

The above computations can be put in an unified context, as follows. Given a simplicial complex L on n vertices, define the associated *toric complex*, T_L, as the subcomplex of the n-torus, obtained by deleting the cells corresponding to the missing simplices of L, i.e.,

$$(6) \qquad T_L = \bigcup_{\sigma \in L} T^\sigma, \quad \text{where } T^\sigma = \{x \in T^n \mid x_i = * \text{ if } i \notin \sigma\}.$$

This construction behaves well with respect to simplicial joins: $T_{L * L'} = T_L \times T_{L'}$.

Let $\Gamma = (\mathsf{V}, \mathsf{E})$ be the graph with vertex set V the 0-cells of L, and edge set E the 1-cells of L. The fundamental group of the toric complex T_L is the *right-angled Artin group*

$$(7) \qquad G_\Gamma = \langle v \in \mathsf{V} \mid vw = wv \text{ if } \{v, w\} \in \mathsf{E} \rangle.$$

Groups of this sort interpolate between $G_\Gamma = \mathbb{Z}^n$ in case Γ is the complete graph K_n, and $G_\Gamma = F_n$ in case Γ is the discrete graph $\overline{K}_n$. Evidently, this class of groups is closed under direct products: $G_\Gamma \times G_{\Gamma'} = G_{\Gamma * \Gamma'}$.

Given a right-angled Artin group G_Γ, identify the character group $\text{Hom}(G_\Gamma, \mathbb{k}^\times)$ with the algebraic torus $(\mathbb{k}^\times)^{\mathsf{V}} := (\mathbb{k}^\times)^n$. For each subset $\mathsf{W} \subseteq \mathsf{V}$, let $(\mathbb{k}^\times)^{\mathsf{W}} \subseteq (\mathbb{k}^\times)^{\mathsf{V}}$ be the corresponding subtorus; in particular, $(\mathbb{k}^\times)^\emptyset = \{1\}$.

Theorem 2.4 ([72]). *With notation as above,*

$$(8) \qquad \mathcal{V}_d^i(T_L, \mathbb{k}) = \bigcup_{\substack{\mathsf{W} \subseteq \mathsf{V} \\ \sum_{\sigma \in L_{\mathsf{V} \setminus \mathsf{W}}} \dim_\mathbb{k} \widetilde{H}_{i-1-|\sigma|}(\text{lk}_{L_\mathsf{W}}(\sigma), \mathbb{k}) \geq d}} (\mathbb{k}^\times)^{\mathsf{W}},$$

where L_W is the subcomplex induced by L on W, and $\text{lk}_K(\sigma)$ is the link of a simplex σ in a subcomplex $K \subseteq L$.

A classifying space for the group G_Γ is the toric complex T_{Δ_Γ}, where Δ_Γ is the *flag complex* of Γ, i.e., the maximal simplicial complex with 1-skeleton equal to the graph Γ. Thus, for $i = d = 1$, formula (8) yields:

$$(9) \qquad \mathcal{V}_1(G_\Gamma, \mathbb{k}) = \bigcup_{\substack{\mathsf{W} \subseteq \mathsf{V} \\ \Gamma_\mathsf{W} \text{ disconnected}}} (\mathbb{k}^\times)^{\mathsf{W}}.$$

2.4. Tangent cones. We now return to the general situation from §2.2. In the sequel, we will be interested in approximating the characteristic varieties $\mathcal{V}_d^i(X, \mathbb{C})$ by their tangent cones (or exponential versions thereof) at the origin, that is, $1 \in \mathrm{Hom}(\pi_1(X, x_0), \mathbb{C}^\times)$. We conclude this section with a review of these constructions.

Let G be a finitely generated group. Its character group, $\mathrm{Hom}(G, \mathbb{C}^\times)$, may be identified with the cohomology group $H^1(G, \mathbb{C}^\times)$, while the group $\mathrm{Hom}(G, \mathbb{C})$ may be identified with $H^1(G, \mathbb{C})$. The exponential map $\mathbb{C} \to \mathbb{C}^\times$, $z \mapsto e^z$ is a group homomorphism. As such, it defines a coefficient homomorphism, $\exp \colon H^1(G, \mathbb{C}) \to H^1(G, \mathbb{C}^\times)$.

Now let W be a Zariski closed subset of $\mathrm{Hom}(G, \mathbb{C}^\times)$. The *tangent cone* of W at 1 is the subset of $H^1(G, \mathbb{C}) = \mathbb{C}^n$, where $n = b_1(G)$, defined as follows. Let J be the ideal in the ring of analytic functions $\mathbb{C}\{z_1, \ldots, z_n\}$ defining the germ of W at 1, and let $\mathrm{in}(J)$ be the ideal in the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$ spanned by the initial forms of non-zero elements of J. Then

$$\text{(10)} \qquad TC_1(W) = V(\mathrm{in}(J)).$$

On the other hand, the *exponential tangent cone* of W at 1 is the set

$$\text{(11)} \qquad \tau_1(W) = \{z \in H^1(G, \mathbb{C}) \mid \exp(tz) \in W, \text{ for all } t \in \mathbb{C}\}.$$

It is readily seen that both $TC_1(W)$ and $\tau_1(W)$ are homogeneous subvarieties of $H^1(G, \mathbb{C})$, depending only on the analytic germ of W at the identity. In particular, $TC_1(W)$ and $\tau_1(W)$ are non-empty if and only if 1 belongs to W.

Lemma 2.5 ([30])**.** *For any subvariety $W \subseteq H^1(G, \mathbb{C}^\times)$, the exponential tangent cone $\tau_1(W)$ is a finite union of rationally defined linear subspaces of $H^1(G, \mathbb{C})$.*

Let us describe these subspaces explicitly. Clearly, τ_1 commutes with intersections, so it is enough to consider the case $W = V(f)$, where $f = \sum_{u \in S} c_u t_1^{u_1} \cdots t_n^{u_n}$ is a non-zero Laurent polynomial, with support $S \subseteq \mathbb{Z}^n$. We may assume $f(1) = 0$, for otherwise, $\tau_1(W) = \emptyset$. Let $\mathcal{P}$ be the set of partitions $p = S_1 \coprod \cdots \coprod S_r$ of S, having the property that $\sum_{u \in S_i} c_u = 0$, for all $1 \leq i \leq r$. For each such partition, let $L(p)$ be the (rational) linear subspace consisting of all points $z \in \mathbb{C}^n$ for which the dot product $\langle u - v, z \rangle$ vanishes, for all $u, v \in S_i$ and all $1 \leq i \leq r$. Then, as shown in [30, Lemma 4.3],

$$\text{(12)} \qquad \tau_1(W) = \bigcup_{p \in \mathcal{P}} L(p).$$

By [74, Proposition 4.4], the following inclusion holds for any subvariety $W \subseteq H^1(G, \mathbb{C}^\times)$:

$$\text{(13)} \qquad \tau_1(W) \subseteq TC_1(W).$$

If all irreducible components of W containing 1 are subtori, then clearly $\tau_1(W) = TC_1(W)$, but in general the two types of tangent cones differ.

3. Homology of abelian covers

Much on the initial motivation for studying the characteristic varieties of a space comes from the precise information they give about the homology of its regular abelian covers. We now describe some of the ways in which this is achieved.

3.1. Finite abelian covers. Work of Libgober [**55**], Hironaka [**50**, **51**], Sarnak and Adams [**79**], and Sakuma [**78**] revealed the varied and fruitful connections between the characteristic varieties, the Alexander invariants (see §4 below), and the Betti numbers of finite abelian covers. Let us summarize two of those results, in a somewhat stronger form, following the treatment from [**65**] and [**83**].

As before, let X be a connected CW-complex with finite 1-skeleton, and denote by G its fundamental group. Let $p\colon Y \to X$ be a regular, n-fold cyclic cover, with classifying map $\lambda\colon G \twoheadrightarrow \mathbb{Z}_n$. Fix an algebraically closed field $\Bbbk$, and assume $\operatorname{char}\Bbbk \nmid n$. Then, the homomorphism $\iota\colon \mathbb{Z}_n \to \Bbbk^\times$ which sends a generator of $\mathbb{Z}_n$ to a primitive n-th root of unity in $\Bbbk$ is an injection. For each integer $j > 0$, define a character $(\iota \circ \lambda)^j\colon G \to \Bbbk^\times$ by $(\iota \circ \lambda)^j(g) = \iota(\lambda(g))^j$. Proceeding as in the proof of Theorem 6.1 from [**65**], we obtain the following slightly more general result.

Theorem 3.1 ([**65**]). *Let $\lambda\colon G \twoheadrightarrow \mathbb{Z}_n$ be an epimorphism, and let $Y \to X$ be the corresponding regular cover. If $\bar{\Bbbk} = \Bbbk$ and $\operatorname{char}\Bbbk \nmid n$, then*

$$(14) \qquad \dim_\Bbbk H_1(Y, \Bbbk) = \dim_\Bbbk H_1(X, \Bbbk) + \sum_{1 \neq k \mid n} \varphi(k) \operatorname{depth}_\Bbbk \big((\iota \circ \lambda)^{n/k}\big),$$

where φ is the Euler totient function.

In the same vein, for each $n > 1$, let X_n be the n-th congruence cover of X, i.e., the regular cover determined by the canonical projection $\pi_1(X, x_0) \twoheadrightarrow H_1(X, \mathbb{Z}_n)$. Then, a mild generalization of Theorem 5.2 from [**83**] yields the following formula for the first Betti number of such a cover:

$$(15) \qquad b_1(X_n) = b_1(X) + \sum_{\rho} \operatorname{depth}_\mathbb{C}(\rho),$$

where the sum is taken over all characters $\rho \in \operatorname{Hom}(G, \mathbb{C}^\times)$ of order exactly equal to n.

3.2. Free abelian covers. It turns out that the characteristic varieties also control the homological finiteness properties of (regular) free abelian covers.

Let $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$ ($r > 0$) be an epimorphism, and let $X^\nu \to X$ be the corresponding cover. Denote by $\nu^*\colon \operatorname{Hom}(\mathbb{Z}^r, \Bbbk^\times) \to \operatorname{Hom}(G, \Bbbk^\times)$ the induced homomorphism between character groups. When $r = 1$, denote by $\nu_\Bbbk \in H^1(X, \Bbbk)$ the corresponding cohomology class.

Theorem 3.2 ([**34**], [**74**]). *Suppose X has finite k-skeleton.*

   (i) $\sum_{i=0}^k \dim_\Bbbk H_i(X^\nu, \Bbbk) < \infty \iff \operatorname{im}(\nu^*) \cap \big(\bigcup_{i=0}^k \mathcal{V}_1^i(X, \Bbbk)\big)$ *is finite.*
   (ii) *If $r = 1$, then $\sum_{i=0}^k \dim_\mathbb{C} H_i(X^\nu, \mathbb{C}) < \infty \iff \nu_\mathbb{C} \notin \bigcup_{i=0}^k \tau_1\big(\mathcal{V}_1^i(X, \mathbb{C})\big)$.*

Part (i) was proved by Dwyer and Fried in [**34**], and reinterpreted in this context in [**74**, Corollary 6.2], while part (ii) was proved in [**74**, Theorem 6.5].

An epimorphism $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$ as above gives rise to a map $\bar{\nu}\colon H_1(X, \mathbb{Q}) \twoheadrightarrow \mathbb{Q}^r$, which may be viewed as an element in the Grassmanian $\operatorname{Gr}_r(H^1(X, \mathbb{Q}))$ of r planes in the vector space $H^1(X, \mathbb{Q})$. Following [**34**], consider the set

$$(16) \qquad \Omega_r^k(X) = \Big\{\bar{\nu} \in \operatorname{Gr}_r(H^1(X, \mathbb{Q})) \mid \sum_{i=0}^k \dim_\mathbb{C} H_i(X^\nu, \mathbb{C}) < \infty\Big\}.$$

By Theorem 3.2(ii) and Lemma 2.5, $\Omega_1^k(X)$ is the complement of a finite union of projective subspaces; in particular, it is an open set. For $r > 1$, though, $\Omega_r^k(X)$ is not necessarily open, as the following example of Dwyer and Fried shows.

Example 3.3 ([34]). Let $Y = T^3 \vee S^2$. Then $\pi_1(Y) = H$, a free abelian group on generators x_1, x_2, x_3, and $\pi_2(Y) = \mathbb{Z}H$, a free module generated by the inclusion of S^2 in Y. Let $f \colon S^2 \to Y$ represent the element $x_1 - x_2 + 1$ of $\pi_2(Y)$, and attach a 3-cell to Y along f to obtain a CW-complex $X = Y \cup_f D^3$, with $\pi_1(X) = H$ and $\pi_2(X) = \mathbb{Z}H/(x_1 - x_2 + 1)$. Identifying $\operatorname{Hom}(H, \mathbb{C}) = (\mathbb{C}^\times)^3$, we have $\mathcal{V}_1^1(X) = \{1\}$ and $\mathcal{V}_1^2(X) = \{z \in (\mathbb{C}^\times)^3 \mid z_1 - z_2 + 1 = 0\}$.

Consider an algebraic 2-torus T in $(\mathbb{C}^\times)^3$, given by an equation of the form $\{z_1^{a_1} z_2^{a_2} z_3^{a_3} = 1\}$, for some $a_i \in \mathbb{Z}$. It is readily checked that the subvariety $T \cap \mathcal{V}_1^1(X)$ is either the empty set (this happens precisely when $T = \{z_1 z_2^{-1} = 1\}$ or $T = \{z_2 = 1\}$), or is 1-dimensional. Thus, the locus in the Grassmannian of 2-planes in $H^1(X, \mathbb{Q}) = \mathbb{Q}^3$ giving rise to algebraic 2-tori in $(\mathbb{C}^\times)^3$ having finite intersection with $\mathcal{V}_1^2(X)$ consists of two points. It follows that exactly two $\mathbb{Z}^2$-covers of X have finite Betti numbers. In particular, the set $\Omega_2^2(X)$ is not open in $\operatorname{Gr}_2(H^1(X, \mathbb{Q})) = \mathbb{QP}^2$, even for the usual topology on this projective plane.

For an in-depth discussion of the interplay between the geometry of the cohomology jumping loci and the homological finiteness properties of free abelian covers, we refer to [85] and [86].

4. Alexander invariants

In this section, we discuss the various Alexander-type invariants associated to a space, with special emphasis on the Alexander polynomial and the Alexander varieties, and how these objects relate to the characteristic varieties.

4.1. Alexander modules. As before, let X be a connected CW-complex, with a unique 0-cell, x_0, which we take as the basepoint, and with finitely many 1-cells. Let $G = \pi_1(X, x_0)$ be the fundamental group, and let

$$(17) \qquad\qquad H = H_1(G, \mathbb{Z})/\text{torsion} \cong \mathbb{Z}^{b_1(G)}$$

be its maximal torsion-free abelian quotient. The canonical projection, $\mathrm{ab} \colon G \twoheadrightarrow H$, determines a regular cover, $\pi \colon X' \to X$. Set $F = \pi^{-1}(x_0)$. The exact sequence of the pair (X', F) yields an exact sequence of $\mathbb{Z}H$-modules,

$$(18) \qquad 0 \longrightarrow H_1(X', \mathbb{Z}) \longrightarrow H_1(X', F, \mathbb{Z}) \longrightarrow H_0(F, \mathbb{Z}) \longrightarrow H_0(X', \mathbb{Z}) \longrightarrow 0 .$$

The module $H_1(X', \mathbb{Z})$ is called the *(first) Alexander invariant* of X, while $H_1(X', F, \mathbb{Z})$ is called the *Alexander module* of X. Clearly, these two $\mathbb{Z}H$-modules depend only on the fundamental group $G = \pi_1(X, x_0)$, so we denote them by B_G and A_G, respectively. Identifying the kernel of the map $H_0(F, \mathbb{Z}) \to H_0(X', \mathbb{Z})$ with the augmentation ideal, $I_H = \ker\big(\epsilon \colon \mathbb{Z}H \to \mathbb{Z}\big)$, one extracts from (18) the exact sequence $0 \to B_G \to A_G \to I_H \to 0$.

4.2. Fox derivatives and the Alexander matrix. The free differential calculus of R. Fox [47, 48] yields an efficient algorithm for computing the Alexander module of a finitely generated group G. For all practical purposes, we may assume G is finitely presented; as explained in [29, §2.6], there is no real loss of generality in doing that.

Let F_q be the free group with generators $x_1, \ldots, x_q$. For each $1 \le j \le q$, there is a linear operator $\partial_j = \partial/\partial x_j \colon \mathbb{Z}F_q \to \mathbb{Z}F_q$, known as the j-th Fox derivative, uniquely determined by the following rules: $\partial_j(1) = 0$, $\partial_j(x_i) = \delta_{ij}$, and $\partial_j(uv) = \partial_j(u)\epsilon(v) + u\partial_j(v)$.

Next, let $G = \langle x_1, \ldots, x_q \mid r_1, \ldots, r_m \rangle$ be a finite presentation for our group, and let $\phi \colon F_q \to G$ be the presenting homomorphism. Define the *Jacobian matrix* of (the given presentation of) G as

$$(19) \qquad\qquad J_G = \left(\partial_j r_i\right)^\phi \colon \mathbb{Z}G^m \to \mathbb{Z}G^q.$$

As noted by Fox (see [**65**, Theorem 4.1] for a short proof), the matrix J_G determines the first homology group of any finite-index subgroup of G.

Proposition 4.1 ([**48**])**.** *Let $K < G$ be a subgroup of index $k < \infty$, let $\sigma \colon G \to \mathrm{Sym}(G/K) \cong S_k$ be the coset representation, and let $\pi \colon S_k \to \mathrm{GL}(k, \mathbb{Z})$ be the permutation representation. Then $J_G^{\pi \circ \sigma}$ is a presentation matrix for the abelian group $H_1(K) \oplus \mathbb{Z}^{k-1}$.*

Define now the *Alexander matrix* of the group G as the (torsion-free) abelianization of the Fox Jacobian matrix of G,

$$(20) \qquad\qquad \Phi_G = J_G^{\mathrm{ab}} \colon \mathbb{Z}H^m \to \mathbb{Z}H^q.$$

Proposition 4.2 ([**47**])**.** *The Alexander matrix Φ_G is a presentation matrix for the Alexander module A_G.*

4.3. Elementary ideals. Before proceeding, we need to recall some basic notions from commutative algebra. Let R be a commutative ring with unit. Assume R is Noetherian and a unique factorization domain. Let M be a finitely generated R-module. Then M admits a finite presentation, of the form

$$(21) \qquad\qquad R^m \xrightarrow{\ \Phi\ } R^q \longrightarrow M \longrightarrow 0 \ .$$

The *i-th elementary ideal* of M, denoted $E_i(M)$, is the ideal generated by the minors of size $q - i$ of the $q \times m$ matrix Φ, with the convention that $E_i(M) = R$ if $i \ge q$, and $E_i(M) = 0$ if $q - i > m$. It is a standard exercise to show that $E_i(M)$ does not depend on the choice of presentation (21). Clearly, $E_i(M) \subset E_{i+1}(M)$, for all $i \ge 0$. The annihilator ideal, $\mathrm{ann}(M)$, contains the ideal of maximal minors, $E_0(M)$, and they both have the same radical.

Let $\Delta_i(M)$ be a generator of the smallest principal ideal in R containing $E_i(M)$, i.e., the greatest common divisor of all elements of $E_i(M)$. As such, $\Delta_i(M)$ is well-defined only up to units in R. (If two elements Δ, Δ' in R generate the same principal ideal, that is, $\Delta = u\Delta'$, for some unit $u \in R^*$, we shall write $\Delta \doteq \Delta'$.) Note that $\Delta_{i+1}(M)$ divides $\Delta_i(M)$, for all $i \ge 0$.

4.4. Alexander polynomial. As before, let G be a finitely generated group, and let $H = \mathrm{ab}(G)$ be its maximal torsion-free abelian quotient. It is readily seen that the group ring $\mathbb{Z}H$ is a (commutative) Noetherian ring, and a unique factorization domain.

Definition 4.3. The *Alexander polynomial* of the group G is the greatest common divisor of all elements in the first elementary ideal of the Alexander module of G,

$$(22) \qquad\qquad \Delta_G = \Delta_1(A_G) = \gcd(E_1(A_G)) \in \mathbb{Z}H.$$

From the discussion in §4.3, it follows that Δ_G depends only on G, modulo units in $\mathbb{Z}H$.

Now suppose G admits a finite presentation, with q generators and m relators. Fix a basis $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ for the free abelian group $H = \operatorname{ab}(G)$. This identifies the group ring $\mathbb{Z}H$ with the Laurent polynomial ring $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. In this fashion, the Alexander polynomial of G may be viewed as a Laurent polynomial in n variables, well-defined up to monomials of the form $u = \pm t_1^{\nu_1} \cdots t_n^{\nu_n}$. By Proposition 4.2, Δ_G is the greatest common divisor of the minors of size $q - 1$ of the Alexander matrix, $\Phi_G \colon \Lambda^m \to \Lambda^q$.

Remark 4.4. Note that the Alexander polynomial, when viewed as an element of Λ, depends on the choice of basis α for $H = \mathbb{Z}^n$, even though we suppress that dependence from the notation. If α' is another choice of basis, then the corresponding polynomial, Δ'_G, is obtained from Δ_G by applying the automorphism of the ring $\Lambda = \mathbb{Z}\mathbb{Z}^n$ induced by the linear automorphism of $\mathbb{Z}^n$ taking α to α'. Nevertheless, various features of the Alexander polynomial, such as the constancy of Δ_G, or the dimension of its Newton polytope, are unaffected by the choice of basis for H.

4.5. Alexander varieties. Let X be CW-complex as in §4.1, with fundamental group $G = \pi_1(X, x_0)$, and let $X' \to X$ be the maximal abelian cover, defined by the projection $\operatorname{ab} \colon G \twoheadrightarrow H$. Fix a field $\Bbbk$, and let $\operatorname{Hom}(G, \Bbbk^\times)^0$ be the identity component of the algebraic group $\operatorname{Hom}(G, \Bbbk^\times)$. Clearly, the map ab induces an isomorphism $\operatorname{ab}^* \colon \operatorname{Hom}(H, \Bbbk^\times) \xrightarrow{\cong} \operatorname{Hom}(G, \Bbbk^\times)^0$.

Now consider the i-th Alexander invariant of X, defined as the homology group $H_i(X', \Bbbk)$, viewed as a module over the group ring $\Bbbk H$. The support loci of the elementary ideals of this finitely generated $\Bbbk H$-module define a filtration of the character group $\operatorname{Spec} \Bbbk H = \operatorname{Hom}(H, \Bbbk^\times)$.

Definition 4.5. The *Alexander varieties* of X (over $\Bbbk$) are the subvarieties of $\operatorname{Hom}(G, \Bbbk^\times)^0$ given by

$$\mathcal{W}_d^i(X, \Bbbk) = \operatorname{ab}^*(V(E_{d-1}(H_i(X', \Bbbk)))).$$

In particular, $\mathcal{W}_1^i(X, \Bbbk) = \operatorname{ab}^*(V(\operatorname{ann} H_i(X', \Bbbk)))$.

In degree $i = 1$, these varieties depend only on the group $G = \pi_1(X, x_0)$. Suppose G admits a finite presentation, with, say, q generators, and identify the group $\operatorname{Hom}(G, \Bbbk^\times)^0$ with the algebraic torus $(\Bbbk^\times)^{b_1(G)}$. The varieties $\mathcal{W}_d(G, \Bbbk) = \mathcal{W}_d^1(X, \Bbbk)$ can then be described as the subvarieties of this torus, defined by the vanishing of all minors of size $q - d$ of the Alexander matrix Φ_G.

4.6. Alexander varieties and characteristic varieties. It has been known since the work of Hironaka [51] that the characteristic and Alexander varieties of a space are closely related. As shown in [74, Theorem 3.6], there is an exact match between the filtrations defined on the identity component of the character variety by these subvarieties, at least for depth $d = 1$.

Theorem 4.6 ([74]). *For all $q \geq 0$,*

$$(23) \qquad \operatorname{Hom}(G, \Bbbk^\times)^0 \cap \left(\bigcup_{i=0}^q \mathcal{V}_1^i(X, \Bbbk) \right) = \bigcup_{i=0}^q \mathcal{W}_1^i(X, \Bbbk).$$

In homological degree 1, there is a more precise comparison, valid for arbitrary depths. The following result was proved in [51], and further extended in [65] and [29].

Proposition 4.7. *Let $\rho\colon H \to \Bbbk^{\times}$ be a non-trivial character. Then, for all $d \geq 1$,*

$$\mathrm{ab}^*(\rho) \in \mathcal{V}_d(G, \Bbbk) \Longleftrightarrow \rho \in V(E_d(A_G \otimes \Bbbk)) \Longleftrightarrow \rho \in V(E_{d-1}(B_G \otimes \Bbbk)).$$

We now isolate a class of groups G for which all characteristic varieties $\mathcal{V}_d(G, \Bbbk)$ are determined by the Alexander matrix Φ_G, even at the trivial character.

Definition 4.8. A group G is said to be a *commutator-relators* group if it admits a presentation of the form $G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$, with each relator r_i belonging to the commutator subgroup of $F_n = \langle x_1, \ldots, x_n \rangle$.

For a commutator-relators group G as above, the group $H = H_1(G, \mathbb{Z})$ is free abelian of rank n, and comes endowed with a preferred basis, $\mathrm{ab}(x_1), \ldots, \mathrm{ab}(x_n)$. This allows us to identify in a standard way $\mathrm{Hom}(G, \Bbbk^{\times})$ with $(\Bbbk^{\times})^n$. Let $\Phi_G \colon \Lambda^m \to \Lambda^n$ be the Alexander matrix of G, and let $\Phi_G(t) \colon \Bbbk^m \to \Bbbk^n$ be its evaluation at a point $t = (t_1, \ldots, t_n)$ in $(\Bbbk^{\times})^n$. A routine computation with Fox derivatives shows that $\Phi_G(1)$ is the zero matrix. The next result then follows from Propositions 4.2 and 4.7.

Proposition 4.9. *Let G be a commutator-relators group, with $b_1(G) = n$. Then $\mathcal{V}_d(G, \Bbbk) = \{t \in (\Bbbk^{\times})^n \mid \mathrm{rank}\, \Phi_G(t) < n - d\}$.*

4.7. Alexander polynomial and characteristic varieties. Let $\mathcal{V}_1(G) = \mathcal{V}_1(G, \mathbb{C})$ be the depth 1 characteristic variety of G, with coefficients in $\mathbb{C}$. Denote by $\check{\mathcal{V}}_1(G)$ the union of all codimension-one irreducible components of $\mathcal{V}_1(G) \cap \mathrm{Hom}(G, \mathbb{C}^{\times})^0$. The Alexander polynomial Δ_G defines a hypersurface, $V(\Delta_G)$, in the complex algebraic torus $\mathrm{Hom}(G, \mathbb{C}^{\times})^0$. The next theorem details the relationship between $\check{\mathcal{V}}_1(G)$ and $V(\Delta_G)$.

Theorem 4.10 ([29]). *For a finitely generated group G, the following hold:*

(i) $\Delta_G = 0$ *if and only if* $\mathrm{Hom}(G, \mathbb{C}^{\times})^0 \subseteq \mathcal{V}_1(G)$. *In this case,* $\check{\mathcal{V}}_1(G) = \emptyset$.
(ii) *If* $b_1(G) \geq 1$ *and* $\Delta_G \neq 0$, *then*

$$\check{\mathcal{V}}_1(G) = \begin{cases} V(\Delta_G) & \text{if } b_1(G) > 1 \\ V(\Delta_G) \coprod \{1\} & \text{if } b_1(G) = 1. \end{cases}$$

(iii) *If* $b_1(G) \geq 2$, *then* $\check{\mathcal{V}}_1(G) = \emptyset$ *if and only if* $\Delta_G \doteq \mathrm{const}$.

In particular, if Δ_G does not vanish identically, then $\check{\mathcal{V}}_1(G) \setminus \{1\} = V(\Delta_G) \setminus \{1\}$. Moreover, under all circumstances,

(24) $$\mathcal{V}_1(G) \setminus \{1\} \supseteq V(\Delta_G) \setminus \{1\}.$$

In general, we cannot expect a perfect match between $\mathcal{V}_1(G)$ and $V(\Delta_G)$, as the following example from knot theory illustrates.

Example 4.11. Let K be a non-trivial knot in the 3-sphere, with complement $X = S^3 \setminus K$. The knot group, $G = \pi_1(X)$, has abelianization $\mathbb{Z}$, and thus $1 \in \mathcal{V}_1(G)$. On the other hand, the Alexander polynomial of the knot, $\Delta_G \in \mathbb{Z}[t^{\pm 1}]$, satisfies $\Delta_G(1) = \pm 1$; thus, $1 \notin V(\Delta_G)$. Nevertheless, by Theorem 4.10(ii), $\mathcal{V}_1(G) \setminus \{1\}$ equals $V(\Delta_G)$, the set of roots of the Alexander polynomial.

As a further application of Theorem 4.10, we consider the Alexander polynomial Δ_{G_Γ} associated to a right-angled Artin group, and ask: for which graphs Γ is Δ_{G_Γ} constant? It is easy to see that $\Delta_{F_n} = 0$, for $n \geq 1$, while $\Delta_{\mathbb{Z}^n} \doteq 1$, for $n > 1$. To handle the general case, we need to recall a definition from graph theory.

Definition 4.12. The *connectivity* of a graph $\Gamma = (\mathsf{V}, \mathsf{E})$, denoted $\kappa(\Gamma)$, is the maximum integer r so that, for any subset $\mathsf{W} \subset \mathsf{V}$ with $|\mathsf{W}| < r$, the induced subgraph on $\mathsf{V} \setminus \mathsf{W}$ is connected.

Proposition 4.13. *A right-angled Artin group G_Γ has non-constant Alexander polynomial if and only if the graph Γ has connectivity 1:*

$$\Delta_{G_\Gamma} \not\equiv \mathrm{const} \iff \kappa(\Gamma) = 1.$$

PROOF. Recall from formula (9) that $\mathcal{V}_1(G_\Gamma)$ consists of coordinate subspaces $(\mathbb{C}^\times)^{\mathsf{W}}$, indexed by (maximal) subsets $\mathsf{W} \subset \mathsf{V}$ such that Γ_{W} is disconnected. Thus, $\check{\mathcal{V}}_1(G_\Gamma)$ is non-empty if and only if Γ is connected and has a cut point, i.e., $\kappa(\Gamma) = 1$.

If Γ has just 1 vertex, then $\kappa(\Gamma) = 0$; on the other hand, $G_\Gamma = \mathbb{Z}$, and so $\Delta_{G_\Gamma} = 0$. For all other graphs, $b_1(G_\Gamma) \geq 2$, and Theorem 4.10(iii) yields the desired conclusion. $\square$

4.8. Almost principal Alexander ideals. We conclude this section with a class of groups G for which the Alexander polynomial Δ_G may be used to inform in a more precise fashion on the characteristic varieties $\mathcal{V}_d(G)$. We start with a definition from [29], inspired by work of Eisenbud–Neumann [35] and McMullen [66].

Definition 4.14. Let G be a finitely generated group, and set $H = \mathrm{ab}(G)$. We say that the Alexander ideal $E_1(A_G) \subset \mathbb{Z}H$ is *almost principal* if there exists an integer $q \geq 0$ such that the following inclusion holds in $\mathbb{C}H$:

(25) $$I_H^q \cdot (\Delta_G) \subseteq E_1(A_G) \otimes \mathbb{C}.$$

For this class of groups, the Alexander polynomial determines to a large extent the depth 1 characteristic variety.

Proposition 4.15. *Suppose the Alexander ideal $E_1(A_G)$ is almost principal. Then:*
   (i) $\left(\mathcal{V}_1(G) \cap \mathrm{Hom}(G, \mathbb{C}^\times)^0 \right) \setminus \{1\} = V(\Delta_G) \setminus \{1\}$.
   (ii) *If, moreover, $H_1(G, \mathbb{Z})$ is torsion-free, then $\mathcal{V}_1(G) \setminus \{1\} = V(\Delta_G) \setminus \{1\}$.*

PROOF. Part (i) follows from (24) and (25). Part (ii) is now obvious. $\square$

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Finally, let us remark on the connection between the multiplicities of the factors of Δ_G and the higher-depth characteristic varieties of G. Identify $\mathrm{Hom}(G, \mathbb{C}^\times)^0 = (\mathbb{C}^\times)^n$, where $n = b_1(G)$. For a character $\rho \in (\mathbb{C}^\times)^n$, and a Laurent polynomial $f \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, denote by $\nu_\rho(f)$ the order of vanishing of the germ of f at ρ.

Theorem 4.16 ([29]). *Suppose the Alexander ideal of G is almost principal, and let $\Delta_G \doteq c f_1^{\mu_1} \cdots f_s^{\mu_s}$ be the decomposition into irreducible factors of the Alexander polynomial. Then, $\dim_{\mathbb{C}} H_1(G; \mathbb{C}_\rho) \leq \sum_{j=1}^s \mu_j \cdot \nu_\rho(f_j)$, for all $\rho \in \mathrm{Hom}(G, \mathbb{C}^\times)^0 \setminus \{1\}$.*

If the upper bound from Theorem 4.16 is attained for every nontrivial character $\rho \colon G \to \mathbb{C}^\times$, then clearly the Alexander polynomial Δ_G determines the characteristic varieties $\mathcal{V}_d(G)$, for all $d \geq 1$, at least away from 1.

5. Resonance varieties

We now look at the jumping loci associated to the cohomology ring of a space, and how they relate to the Alexander matrix and to the characteristic varieties.

5.1. Jump loci for the Aomoto complex. As before, let X be a connected CW-complex with finite k-skeleton, for some $k \geq 1$. Also, let $\Bbbk$ be a field; if char $\Bbbk = 2$, assume additionally that $H_1(X, \mathbb{Z})$ has no 2-torsion.

Consider the cohomology algebra $A = H^*(X, \Bbbk)$, with graded ranks the $\Bbbk$-Betti numbers, $b_i = \dim_\Bbbk A^i$. For each $a \in A^1$, we have $a^2 = 0$, by graded-commutativity of the cup product (and our assumption on the 2-torsion). Thus, right-multiplication by a defines a cochain complex

$$(26) \qquad (A, \cdot a) \colon \ A^0 \xrightarrow{\ a\ } A^1 \xrightarrow{\ a\ } A^2 \longrightarrow \cdots,$$

known as the *Aomoto complex*. Let $\beta_i(A, a) = \dim_\Bbbk H^i(A, \cdot a)$ be the Betti numbers of this complex. The jump loci for the Aomoto-Betti numbers define a natural filtration of the affine space $A^1 = H^1(X, \Bbbk)$.

Definition 5.1. The *resonance varieties* of X (over $\Bbbk$) are the algebraic sets

$$\mathcal{R}^i_d(X, \Bbbk) = \{a \in A^1 \mid \beta_i(A, a) \geq d\},$$

defined for all integers $0 \leq i \leq k$ and $d > 0$.

It is readily seen that each of these sets is a homogeneous algebraic subvariety of $A^1 = \Bbbk^{b_1}$. Indeed, $\beta_i(A, xa) = \beta_i(A, a)$, for all $x \in \Bbbk^\times$, and homogeneity follows. In each degree $i \geq 0$, the resonance varieties provide a descending filtration,

$$(27) \qquad H^1(X, \Bbbk) \supseteq \mathcal{R}^i_1(X, \Bbbk) \supseteq \cdots \supseteq \mathcal{R}^i_{b_i}(X, \Bbbk) \supseteq \mathcal{R}^i_{b_i+1}(X, \Bbbk) = \emptyset.$$

Note that, if $A^i = 0$, then $\mathcal{R}^i_d(X, \Bbbk) = \emptyset$, for all $d > 0$. In degree 0, we have $\mathcal{R}^0_1(X, \Bbbk) = \{0\}$, and $\mathcal{R}^0_d(X, \Bbbk) = \emptyset$, for $d > 1$. In degree 1, the varieties $\mathcal{R}^1_d(X, \Bbbk)$ depend only on the group $G = \pi_1(X)$—in fact, only on the cup-product map $\cup \colon H^1(G, \Bbbk) \wedge H^1(G, \Bbbk) \to H^2(G, \Bbbk)$—so we sometimes denote them by $\mathcal{R}_d(G, \Bbbk)$.

The resonance varieties depend only on the characteristic of the ground field: if $\Bbbk \subseteq \mathbb{K}$ is an extension, then $\mathcal{R}^i_d(X, \Bbbk) = \mathcal{R}^i_d(X, \mathbb{K}) \cap H^1(X, \Bbbk)$. Moreover,

$$(28) \qquad \mathcal{R}^i_1(X_1 \times X_2, \Bbbk) = \bigcup_{p+q=i} \mathcal{R}^p_1(X_1, \Bbbk) \times \mathcal{R}^q_1(X_2, \Bbbk),$$

provided both X_1 and X_2 have finitely many i-cells, see [74, Proposition 13.1].

5.2. Matrix interpretation. An alternate way to compute the degree 1 resonance varieties of the algebra $A = H^*(X, \Bbbk)$ is to realize them as the determinantal varieties of a certain matrix of linear forms. Let us describe this method, following the approach taken in [64].

By definition, an element $a \in A^1$ belongs to $R_d(X, \Bbbk)$ if and only if there exists a subspace $W \subset A^1$ of dimension $d + 1$ such that $a \cup b = 0$, for all $b \in W$. Fix ordered bases, $\{\alpha_1, \ldots, \alpha_{b_1}\}$ for A^1 and $\{\beta_1, \ldots, \beta_{b_2}\}$ for A^2. The multiplication map, $\mu \colon A^1 \otimes A^1 \to A^2$, is then given by

$$(29) \qquad \mu(\alpha_i, \alpha_j) = \sum_{k=1}^{b_2} \mu_{ijk}\, \beta_k,$$

with coefficients $\mu_{ijk} \in \Bbbk$ satisfying $\mu_{jik} = -\mu_{ijk}$. Denote by A_1 the dual $\Bbbk$-vector space to A^1, and identify the symmetric algebra $S = \operatorname{Sym}(A_1)$ with the polynomial ring $\mathbb{C}[x_1, \ldots, x_{b_1}]$, where x_i is the dual of α_i. With this notation, define the *linearized Alexander matrix* of A as

$$(30) \qquad \Theta_A = (\Theta_{kj}) \colon S^{b_2} \to S^{b_1}, \quad \text{where } \Theta_{kj} = \sum_{i=1}^{b_1} \mu_{ijk} x_i.$$

Adapting the proof of [64, Theorem 3.9] to this slightly more general context, we obtain the following result.

Proposition 5.2 ([64]). *With notation as above,*

$$(31) \qquad \mathcal{R}_d(X, \Bbbk) = V(E_d(\Theta_A)).$$

Now let G be a finitely presented group. Set $H = \operatorname{ab}(G)$, and identify $\Lambda = \Bbbk H$ with $\Bbbk[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, where $n = \operatorname{rank} H$. Let $I = I_H$ be the augmentation ideal. The I-adic completion, $\widehat{\Lambda}$, may be identified with the power series ring $\Bbbk[[t_1, \ldots, t_n]]$, while the associated graded ring, $\operatorname{gr}(\widehat{\Lambda})$, may be identified with the polynomial ring $S = \Bbbk[x_1, \ldots, x_n]$, where $x_i = t_1 - 1$.

The next result (adapted from [64, §3.5]) describes the relationship between the Alexander matrix from (20) and the linearized Alexander matrix from (30), thereby justifying *a posteriori* the terminology.

Proposition 5.3 ([64]). *Let $G = \langle x_1, \ldots, x_q \mid r_1, \ldots, r_m \rangle$ be a finitely presented group. Let $\Phi_G \colon \Lambda^m \to \Lambda^q$ be the Alexander matrix, and $\operatorname{gr}(\widehat{\Phi}_G) \colon S^m \to S^q$ its image under the gr functor. Finally, let $A = H^*(X, \Bbbk)$ be the cohomology ring of the presentation 2-complex. Then:*

$$(32) \qquad \operatorname{gr}(\widehat{\Phi}_G) = \Theta_A.$$

5.3. Tangent cone inclusion. The above discussion hints at a relationship between the characteristic varieties $\mathcal{V}_d(X, \Bbbk)$—the determinantal varieties of the Alexander matrix, at least away from 1—and the resonance varieties $\mathcal{R}_d(X, \Bbbk)$—the determinantal varieties of the linearized Alexander matrix. This relationship, explored in the context of hyperplane arrangements in [18], was established in full generality by Libgober in [58], at least for $\Bbbk = \mathbb{C}$.

Theorem 5.4 ([58]). *Let X be a connected CW-complex with finite k-skeleton. Then, for all $i < k$ and all $d > 0$,*

$$(33) \qquad TC_1(\mathcal{V}_d^i(X, \mathbb{C})) \subseteq \mathcal{R}_d^i(X, \mathbb{C}).$$

For many spaces X, equality holds in (33). We illustrate this phenomenon with the class of spaces discussed in §2.3.

Example 5.5. Let L be a simplicial complex on finite vertex set V, and let $X = T_L$ be the associated toric complex. As shown in [72, Theorem 3.8], the resonance varieties $\mathcal{R}_d^i(T_L, \Bbbk)$ are given by the exact same expression as in (8), with the subtorus $(\Bbbk^\times)^\mathsf{W}$ replaced by the coordinate subspace $\Bbbk^\mathsf{W}$. It follows that the exponential map $\exp \colon \mathbb{C}^\mathsf{V} \to (\mathbb{C}^\times)^\mathsf{V}$ restricts to an isomorphism of analytic germs, $\exp \colon (\mathcal{R}_d^i(T_L), 0) \xrightarrow{\simeq} (\mathcal{V}_d^i(T_L), 1)$, for all $i \geq 0$. In particular,

$$(34) \qquad TC_1(\mathcal{V}_d^i(T_L)) = \mathcal{R}_d^i(T_L).$$

On the other hand, as first noted in [65, Remark 10.3], the inclusion from Theorem 5.4 can be strict. We illustrate this point with a much simpler example than the original one.

Example 5.6. Let $M = G_{\mathbb{R}}/G_{\mathbb{Z}}$ be the 3-dimensional Heisenberg nilmanifold, where $G_{\mathbb{R}}$ is the group of real, unipotent 3×3 matrices, and $G_{\mathbb{Z}} = \pi_1(M)$ is the subgroup of integral matrices in $G_{\mathbb{R}}$. A straightforward computation shows that $\mathcal{V}_1(M) = \{1\}$, and thus $TC_1(\mathcal{V}_1(M)) = \{0\}$. On the other hand, $\mathcal{R}_1(M) = \mathbb{C}^2$, since the cup product vanishes on $H^1(M, \mathbb{C})$. Thus,

$$(35) \qquad TC_1(\mathcal{V}_1(M)) \subsetneqq \mathcal{R}_1(M).$$

For more information on the characteristic and resonance varieties of finitely generated nilpotent groups, we refer to Măcinic and Papadima [63] and Măcinic [62].

6. Bieri–Neumann–Strebel–Renz invariants

In this section, we review the definition of the Σ-invariants of a group G (and, more generally, of a space X), and discuss the relation of these invariants to the homology jumping loci.

6.1. Σ-invariants. We start with a definition given by Bieri, Neumann, and Strebel in [8]. Let G be a finitely generated group. Choose a finite set of generators for G, and let $\mathcal{C}(G)$ be the corresponding Cayley graph. Given a homomorphism $\chi \colon G \to \mathbb{R}$, let $\mathcal{C}_\chi(G)$ be the induced subgraph on vertex set $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$. The *BNS invariant* of G is the set

$$(36) \qquad \Sigma^1(G) = \{\chi \in \mathrm{Hom}(G, \mathbb{R}) \setminus \{0\} \mid \mathcal{C}_\chi(G) \text{ is connected}\}.$$

As shown in [8], $\Sigma^1(G)$ is an *open*, conical subset of the vector space $\mathrm{Hom}(G, \mathbb{R}) = H^1(G, \mathbb{R})$; moreover, this set is independent of the choice of generators for G.

In [9], Bieri and Renz extended this definition, as follows. Recall that a group G (or, more generally, a monoid G) is of type FP_k if there is a projective $\mathbb{Z}G$-resolution $P_\bullet \to \mathbb{Z}$, with P_i finitely generated, for all $i \leq k$. In particular, G is of type FP_1 if and only if G is finitely generated. The *BNSR invariants* of G are the sets

$$(37) \qquad \Sigma^q(G, \mathbb{Z}) = \{\chi \in \mathrm{Hom}(G, \mathbb{R}) \setminus \{0\} \mid \text{the monoid } G_\chi \text{ is of type } \mathrm{FP}_q\}.$$

These sets form a descending chain of open subsets of $\mathrm{Hom}(G, \mathbb{R})$, starting at $\Sigma^1(G, \mathbb{Z}) = \Sigma^1(G)$. Moreover, $\Sigma^q(G, \mathbb{Z})$ is non-empty only if G is of type FP_q.

To a large extent, the importance of the Σ-invariants lies in the fact that they control the finiteness properties of kernels of projections to abelian quotients. More precisely, let N be a normal subgroup of G, with G/N abelian. Then, as shown in [8, 9], the group N is of type FP_q if and only if $\{\chi \in \mathrm{Hom}(G, \mathbb{R}) \setminus \{0\} \mid \chi(N) = 0\} \subseteq \Sigma^q(G, \mathbb{Z})$. In particular, the kernel of an epimorphism $\chi \colon G \twoheadrightarrow \mathbb{Z}$ is finitely generated if and only if both χ and $-\chi$ belong to $\Sigma^1(G)$.

6.2. Novikov homology. In [81], Sikorav reinterpreted the BNS invariant of a finitely generated group G in terms of Novikov homology. This interpretation was extended to all BNSR invariants by Bieri [7], leading to a very general definition of the BNSR invariants of a space [45, 74].

The *Novikov–Sikorav completion* of the group ring $\mathbb{Z}G$ with respect to a homomorphism $\chi \colon G \to \mathbb{R}$ consists of all formal sums $\lambda = \sum_i n_i g_i$, with $n_i \in \mathbb{Z}$ and

$g_i \in G$, having the property that, for each $c \in \mathbb{R}$, the set of indices i for which $n_i \neq 0$ and $\chi(g_i) \geq c$ is finite. With the obvious addition and multiplication, the Novikov–Sikorav completion, $\widehat{\mathbb{Z}G}_\chi$, is a ring, containing $\mathbb{Z}G$ as a subring; in particular, $\widehat{\mathbb{Z}G}_\chi$ carries a natural G-module structure. For details, we refer to Farber's book [44].

Now let X be a connected CW-complex with finite 1-skeleton, and let $G = \pi_1(X, x_0)$ be its fundamental group. The *BNSR invariants* of X are the sets

$$(38) \qquad \Sigma^q(X, \mathbb{Z}) = \{\chi \in \mathrm{Hom}(G, \mathbb{R}) \setminus \{0\} \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0, \ \forall\, i \leq q\}.$$

Identifying $\mathrm{Hom}(G, \mathbb{R}) = H^1(G, \mathbb{R}) = H^1(X, \mathbb{R})$, we may view the Σ-invariants of X as subsets of the vector space $H^1(X, \mathbb{R})$. As shown in [7], definitions (37) and (38) agree: If G is a group of type FP_k, then $\Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G, 1), \mathbb{Z})$, for all $q \leq k$.

6.3. Σ-invariants and characteristic varieties. In practice, the BNSR invariants are extremely hard to compute: a complete description of the sets $\Sigma^q(G, \mathbb{Z})$ is known only for some very special classes of groups G, such as one-relator groups and right-angled Artin groups. The following result from [74] gives a computable "upper bound" for the Σ-invariants of a space X or a group G, in terms of the real points on the exponential tangent cones to the respective characteristic varieties (discussed in §2.4).

Theorem 6.1 ([74]). *Let X be a connected CW-complex with finite k-skeleton, for some $k \geq 1$. Then, for each $q \leq k$, the following holds:*

$$(39) \qquad \Sigma^q(X, \mathbb{Z}) \subseteq \left(H^1(X, \mathbb{R}) \cap \bigcup_{i \leq q} \tau_1\big(\mathcal{V}_1^i(X, \mathbb{C})\big)\right)^{\complement}.$$

In particular, for every finitely generated group G,

$$(40) \qquad \Sigma^1(G) \subseteq \big(H^1(G, \mathbb{R}) \cap \tau_1(\mathcal{V}_1(G, \mathbb{C}))\big)^{\complement}.$$

Qualitatively, the above theorem says that each Σ-invariant is contained in the complement of a union of rationally defined subspaces. As noted in [74], bound (39) is sharp. For example, if G is a finitely generated nilpotent group, then $\Sigma^q(G, \mathbb{Z}) = \mathrm{Hom}(G, \mathbb{R}) \setminus \{0\}$, while $\mathcal{V}_1^q(G, \mathbb{C}) = \{1\}$, for all $q \geq 1$. Thus, (39) holds as an equality for $X = K(G, 1)$.

6.4. Σ-invariants and valuations. In [24], Delzant discovered a surprising connection between the BNS invariant $\Sigma^1(G)$, discrete valuations on a field $\Bbbk$, and the first characteristic variety $\mathcal{V}_1^1(G, \Bbbk)$. Delzant's result was extended in [74], as follows.

Theorem 6.2 ([74]). *With notation as above, let $\rho\colon G \to \Bbbk^\times$ be a homomorphism such that $\rho \in \bigcup_{i \leq q} \mathcal{V}_1^i(X, \Bbbk)$, for some $q \leq k$, and let $v\colon \Bbbk^\times \to \mathbb{R}$ be a valuation such that $v \circ \rho \neq 0$. Then $v \circ \rho \notin \Sigma^q(X, \mathbb{Z})$.*

Conversely, let $\chi\colon G \to \mathbb{R}$ be a homomorphism such that $-\chi \in \Sigma^q(X, \mathbb{Z})$, for some $q \leq k$, and let $\xi\colon G \twoheadrightarrow \Gamma$ be the corestriction of χ to its image. Suppose there is a character $\rho\colon \Gamma \to \Bbbk^\times$ which is not an algebraic integer. Then $\rho \circ \xi \notin \bigcup_{i \leq q} \mathcal{V}_1^i(X, \Bbbk)$.

Here, ρ is an *algebraic integer* if there is an element $\Delta = \sum n_\gamma \gamma \in \mathbb{Z}\Gamma$ with $\Delta(\rho) = 0$ and $n_{\gamma_0} = 1$, where γ_0 is the greatest element of $\mathrm{supp}(\Delta)$.

7. Formality properties

In this section, all spaces X will be connected, and homotopy equivalent to a CW-complex with finite 1-skeleton. Likewise, all groups G will be assumed to be finitely generated.

7.1. CDGAs and formality. Fix a ground field $\Bbbk$, of characteristic 0. A commutative differential graded algebra (for short, a CDGA), is a graded, graded-commutative $\Bbbk$-algebra A, endowed with a differential $d_A \colon A \to A$ of degree 1. A CDGA morphism is a quasi-isomorphism if it induces an isomorphism in cohomology. Two CDGAs, A and B, are said to be weakly equivalent if there is a zig-zag of quasi-isomorphisms (going both ways), connecting A to B.

A CDGA is *formal* if it is weakly equivalent to its cohomology algebra, endowed with the zero differential. A CDGA (A, d_A) is *q-formal*, for some $q \geq 1$, if there is a zig-zag of morphisms connecting (A, d_A) to $(H^*(A, d_A), d = 0)$, with each one of these maps inducing an isomorphism in cohomology up to degree q, and a monomorphism in degree $q + 1$.

The best-known formality test is provided by the (higher-order) Massey products. Let us briefly recall the definition of these cohomology operations in the simplest case. Suppose $\alpha_1, \alpha_2, \alpha_3$ are homogeneous elements in $H^*(A)$ such that $\alpha_1 \alpha_2 = \alpha_2 \alpha_3 = 0$. Pick representative cocycles a_i for α_i, and elements $x, y \in A$ such that $dx = a_1 a_2$ and $dy = a_2 a_3$. It is readily checked that $x a_3 - (-1)^{|a_1|} a_1 y$ is a cocyle. The Massey triple product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is the set of cohomology classes of all such cocycles. The image of this set in the quotient ring $H^*(A)/(\alpha_1, \alpha_3)$ is a well-defined element of degree $|\alpha_1| + |\alpha_2| + |\alpha_3| - 1$. The triple product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is *non-vanishing* if this element does not equal 0. If (A, d) is formal, then all Massey products of order 3 (or higher) in $H^*(A)$ vanish. We refer to [23, 87] for proofs and further details.

7.2. Formality of spaces. Given a space X as above, Sullivan [87] constructs an algebra $A_{\mathrm{PL}}(X)$ of polynomial differential forms on X with coefficients in $\Bbbk$, and provides it with a natural CDGA structure. The space X is said to be formal (over $\Bbbk$) if Sullivan's algebra $A_{\mathrm{PL}}(X)$ is formal; likewise, X is q-formal if this CDGA is q-formal. When X is a smooth manifold, and $\Bbbk = \mathbb{R}$ or $\mathbb{C}$, one may replace in the definition the algebra of polynomial forms by de Rham's algebra of differential forms.

Examples of formal spaces include rational cohomology tori, surfaces, toric complexes, compact connected Lie groups, as well as their classifying spaces. On the other hand, the only nilmanifolds which are formal are tori. Formality is preserved under wedges and products of spaces, and connected sums of manifolds.

Whether a space X is 1-formal or not depends only on its fundamental group, $G = \pi_1(X, x_0)$. Indeed, if $f \colon X \to K(G, 1)$ is a classifying map, then the induced homomorphism, $f^* \colon H^i(G, \Bbbk) \to H^i(X, \Bbbk)$, is an isomorphism for $i = 1$ and a monomorphism for $i = 2$.

7.3. 1-formality of groups. In [76], Quillen associates to any group G a pronilpotent, filtered Lie algebra, called the *Malcev completion* of G. Concretely, the group ring $\Bbbk G$ has a Hopf algebra structure, with comultiplication given by $g \mapsto g \otimes g$, and with counit the augmentation map. This Hopf algebra structure naturally extends to the completion of $\Bbbk G$ with respect to powers of the augmentation ideal.

The Malcev completion of G, denoted $\mathfrak{m}(G)$, is the Lie algebra of primitive elements in $\widehat{\Bbbk G}$, equipped with the inverse limit filtration.

For a finitely generated group G, the 1-formality property is equivalent to the quadraticity of $\mathfrak{m}(G)$. More precisely, G is 1-formal if and only if $\mathfrak{m}(G)$ is isomorphic, as a filtered Lie algebra, to the completion with respect to degree of a quadratic Lie algebra.

Examples of 1-formal groups include finitely generated free groups and free abelian groups (more generally, right-angled Artin groups), surface groups, and groups with first Betti number equal to 0 or 1. The 1-formality property is preserved under free products and direct products.

7.4. The tangent cone theorem. The main bridge between the 1-formality property of a group and the nature of its cohomology jumping loci is provided by the following result, which summarizes Theorems A and B from [**30**] in the present setting.

Theorem 7.1 ([**30**])**.** *Let G be a 1-formal group. For each $d > 0$,*

(i) *The exponential map* $\exp\colon H^1(G, \mathbb{C}) \to H^1(G, \mathbb{C}^\times)$ *restricts to an isomorphism of analytic germs,* $\exp\colon (\mathcal{R}_d(G), 0) \xrightarrow{\simeq} (\mathcal{V}_d(G), 1)$.

(ii) *The following "tangent cone formula" holds:* $\tau_1(\mathcal{V}_d(G)) = TC_1(\mathcal{V}_d(G)) = \mathcal{R}_d(G)$.

(iii) *The irreducible components of $\mathcal{R}_d(G)$ are all linear subspaces, defined over $\mathbb{Q}$.*

(iv) *The components of $\mathcal{V}_d(G)$ passing through the origin are all rational subtori of the form $\exp(L)$, with L running through the irreducible components of $\mathcal{R}_d(G)$.*

The tangent cone formula from part (ii) of the theorem can be employed as a non-formality test. We illustrate how this works with a well-known example of a manifold, whose lack of formality is usually detected by means of triple Massey products.

Example 7.2. Let $M = G_\mathbb{R}/G_\mathbb{Z}$ be the Heisenberg nilmanifold. As we saw in Example 5.6, $TC_1(\mathcal{V}_1(G_\mathbb{Z})) \neq \mathcal{R}_1(G_\mathbb{Z})$. Thus, $G_\mathbb{Z}$ is not 1-formal, and hence M is not formal.

Next, we illustrate how the rationality property from part (iii) can also be used to detect non-formality.

Example 7.3 ([**30**])**.** Consider the group $G = \langle x_1, x_2, x_3, x_4 \mid r_1, r_2, r_3 \rangle$, with relators $r_1 = [x_1, x_2]$, $r_2 = [x_1, x_4][x_2^{-2}, x_3]$, and $r_3 = [x_1^{-1}, x_3][x_2, x_4]$. Then $\mathcal{R}_1(G) = \{z \in \mathbb{C}^4 \mid z_1^2 - 2z_2^2 = 0\}$ splits into linear subspaces defined over $\mathbb{R}$, but not over $\mathbb{Q}$. Thus, G is *not* 1-formal.

7.5. Resonance upper bound. We return now to the BNSR invariants. As before, let X be a connected CW-complex with finite k-skeleton, for some $k \geq 1$. Suppose there is an integer $q \leq k$ such that the exponential map induces an isomorphism of analytic germs, $\exp\colon (\mathcal{R}_d^i(G), 0) \xrightarrow{\simeq} (\mathcal{V}_d^i(G), 1)$, for all $i \leq q$. We then get from Theorem 6.1 the following "resonance upper bound" for the q-th Σ-invariant of X:

$$(41) \qquad \Sigma^q(X, \mathbb{Z}) \subseteq \Big(\bigcup_{i \leq q} \mathcal{R}_1^i(X, \mathbb{R}) \Big)^{\complement}.$$

Example 7.4. For a toric complex T_L associated to a finite simplicial complex L, the discussion from Example 5.5, yields $\Sigma^q(T_L, \mathbb{Z}) \subseteq \left(\bigcup_{i \leq q} \mathcal{R}_1^i(T_L, \mathbb{R}) \right)^{\complement}$, for all $q \geq 1$, where

$$(42) \qquad \mathcal{R}_d^i(T_L, \mathbb{R}) = \bigcup_{\substack{\mathsf{W} \subseteq \mathsf{V} \\ \sum_{\sigma \in L_{\mathsf{V} \setminus \mathsf{W}}} \dim_{\Bbbk} \widetilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma), \Bbbk) \geq d}} \mathbb{R}^{\mathsf{W}}.$$

For a right-angled group G_Γ associated to a graph Γ, the BNSR invariants were computed by Meier, Meinert, and Van-Vyck in [**67**]. Comparing these invariants with the resonance varieties of G_Γ as in [**74**, §14.3] reveals that the equality

$$(43) \qquad \Sigma^q(G_\Gamma, \mathbb{Z}) = \left(\bigcup_{i \leq q} \mathcal{R}_1^i(G_\Gamma, \mathbb{R}) \right)^{\complement}$$

holds, provided the following condition is satisfied: for every simplex σ of $\Delta = \Delta_\Gamma$, and for every subset $\mathsf{W} \subseteq \mathsf{V}$ such that $\sigma \cap W = \emptyset$, the homology groups $\widetilde{H}_j(\mathrm{lk}_{\Delta_{\mathsf{W}}}(\sigma), \mathbb{Z})$ are torsion-free, for all $j \leq q - \dim(\sigma) - 2$. This condition is satisfied, for instance, when Γ is a tree, or $q = 1$.

As an immediate application of Theorem 7.1, we have the following corollary.

Corollary 7.5 ([**74**]). *Let G be a 1-formal group. Then*

$$(44) \qquad \Sigma^1(G) \subseteq H^1(G, \mathbb{R}) \setminus \mathcal{R}_1(G, \mathbb{R}).$$

Recall that $\mathcal{R}_1(G, \mathbb{R})$ is a homogeneous subvariety of $H^1(G, \mathbb{R})$. Thus, for equality to hold in (44), the set $\Sigma^1(G)$ must be symmetric about the origin, i.e., $\Sigma^1(G) = -\Sigma^1(G)$. This does not happen, in general.

Example 7.6. Let $G = \langle x_1, x_2 \mid x_1 x_2 x_1^{-1} = x_2^2 \rangle$. Then $H^1(G, \mathbb{R}) = \mathbb{R}$; thus, G is 1-formal, and $\mathcal{R}_1(G, \mathbb{R}) = \{0\}$. On the other hand, it follows from [**9**, Theorem 7.3] that $\Sigma^1(G) = (-\infty, 0)$; in particular, $\Sigma^1(G) \neq -\Sigma^1(G)$ and $\Sigma^1(G) \neq \mathcal{R}_1(G, \mathbb{R})^{\complement}$.

In Example 11.7, we will exhibit a 1-formal group G for which $\Sigma^1(G) = -\Sigma^1(G)$, and yet $\Sigma^1(G) \neq \mathcal{R}_1(G, \mathbb{R})^{\complement}$.

Finally, let us note that inclusion (44) may fail to hold when G is not 1-formal: If $M = G_{\mathbb{R}}/G_{\mathbb{Z}}$ is the Heisenberg nilmanifold, then $\Sigma^1(G) = H^1(G, \mathbb{R}) \setminus \{0\}$, yet $\mathcal{R}_1(G, \mathbb{R})^{\complement} = \emptyset$.

8. Kähler and quasi-Kähler manifolds

In this section, we discuss the Alexander polynomial, cohomology jumping loci, and BNSR invariants of Kähler and quasi-Kähler manifolds and their fundamental groups.

8.1. Kähler manifolds and formality. A compact, connected, complex manifold M is called a *Kähler manifold* if M admits a Hermitian metric h for which the imaginary part $\omega = \Im(h)$ is a closed 2-form. The best known examples are smooth, complex projective varieties.

A finitely presented group G is said to be a *Kähler group* if it can be realized as $G = \pi_1(M)$, where M is a compact Kähler manifold. If M can be chosen to be a smooth, irreducible, complex projective variety, then G is said to be a *projective group*. Both classes of groups are closed under finite direct products. Clearly, every

projective group is a Kähler group, but whether the converse holds is an open problem.

For example, $G = \mathbb{Z}^{2r}$ is the fundamental group of the complex torus $(S^1 \times S^1)^r$, and thus a projective group. Furthermore, all finite groups are projective, by a classical result of J.-P. Serre. For background information on Kähler groups, we refer to the monograph [2].

If M is a compact Kähler manifold, then each cohomology group $H^m(M, \mathbb{Z})$ admits a pure Hodge structure of weight m. That is, the vector space $H^m(M, \mathbb{Z}) \otimes \mathbb{C}$ decomposes as a direct sum $\bigoplus_{p+q=m} H^{p,q}$, with $H^{q,p}$ the complex conjugate of $H^{p,q}$. Hodge theory has strong implications on the topology of compact Kähler manifolds. For example, their odd Betti numbers must be even. Consequently, if G is a Kähler group, then $b_1(G)$ must be even.

A deeper constraint was established by Deligne, Griffiths, Morgan, and Sullivan in [23]. For a compact Kähler manifold M, let d be the exterior derivative, J the complex structure, and $d^c = J^{-1}dJ$. Then the following holds: If α is a form which is closed for both d and d^c, and exact for either d or d^c, then α is exact for dd^c. As a consequence of this "dd^c Lemma," all compact Kähler manifolds are formal; in particular, all Kähler groups are 1-formal.

8.2. Quasi-Kähler manifolds. A manifold X is said to be a *quasi-Kähler manifold* if there is a compact Kähler manifold $\overline{X}$ and a normal-crossings divisor D such that $X = \overline{X} \setminus D$. Smooth, irreducible, quasi-projective complex varieties are examples of quasi-Kähler manifolds.

The notions of quasi-Kähler group and quasi-projective group are defined as above. Both classes of groups are closed under finite direct products. Clearly, every quasi-projective group is a quasi-Kähler group, but again, whether the converse holds is an open problem. Furthermore, every Kähler group is a quasi-Kähler group, but the converse does not hold; for instance, $\mathbb{Z} = \pi_1(\mathbb{C}^\times)$ is a quasi-projective, non-Kähler group.

By a well-known result of Deligne [21, 22], each cohomology group $H = H^k(X, \mathbb{Z})$ of a quasi-projective variety X admits a mixed Hodge structure, that is, an increasing filtration $W_\bullet$ on $H_\mathbb{Q} = H \otimes \mathbb{Q}$, called the weight filtration, and a decreasing filtration $F^\bullet$ on $H_\mathbb{C} = H \otimes \mathbb{C}$, called the Hodge filtration, such that, for each $m \geq 0$, the associated graded piece $\mathrm{gr}_m^W(H_\mathbb{Q})$, together with the filtration induced by $F^\bullet$ on $H_\mathbb{C}$, is a pure Hodge structure of weight m. Similarly, a quasi-Kähler manifold X inherits a mixed Hodge structure from each compactification $\overline{X}$ as above; if X is a smooth, quasi-projective variety, this structure is unique.

For a quasi-Kähler manifold X, the existence of mixed Hodge structures on its cohomology groups puts definite constraints on the topology of X. Not surprisingly, these constraints are weaker than in the Kähler case. For instance, quasi-Kähler groups need not be 1-formal. As an illustration, let X be complex Heisenberg manifold, i.e., the total space of the $\mathbb{C}^\times$-bundle over $\mathbb{C}^\times \times \mathbb{C}^\times$, with Euler number 1; then X is a smooth, quasi-projective variety which fails to be 1-formal.

In this framework, Morgan [69] proved the following result: If X is a smooth, quasi-projective variety with $W_1(H^1(X, \mathbb{C})) = 0$, then X is 1-formal. As noted by Deligne [21, 22], W_1 vanishes whenever X admits a non-singular compactification $\overline{X}$ with $b_1(\overline{X}) = 0$. This happens, for instance, when X is the complement of a hypersurface in $\mathbb{CP}^n$. It follows that fundamental groups of complements of projective hypersurfaces are 1-formal.

Remark 8.1. Let C be an algebraic curve in $\mathbb{CP}^2$, with complement $X = \mathbb{CP}^2 \setminus C$. By the above, the group $\pi_1(X)$ is 1-formal. In fact, as shown by Măcinic [**62**] and Cogolludo–Matei [**13**] (using different methods), a stronger conclusion holds in this case: the space X is formal.

8.3. Characteristic varieties. Foundational results on the structure of the cohomology support loci for local systems on smooth projective varieties, and more generally, on compact Kähler manifolds were obtained by Beauville [**6**], Green–Lazarsfeld [**49**], Simpson [**82**], and Campana [**11**]. A more general result, valid in the quasi-Kähler case, was obtained by Arapura [**3**].

Theorem 8.2 ([**3**]). *Let $X = \overline{X} \setminus D$ be a quasi-Kähler manifold. Then:*

(i) *Each component of $\mathcal{V}_1^1(X)$ is either an isolated unitary character, or of the form $\rho \cdot f^*(H^1(C, \mathbb{C}^\times))$, for some torsion character ρ and some admissible map $f\colon X \to C$.*

(ii) *If either $X = \overline{X}$ or $b_1(\overline{X}) = 0$, then, for all $i \geq 0$ and $d \geq 1$, each component of $\mathcal{V}_d^i(X)$ is of the form $\rho \cdot f^*(H^1(T, \mathbb{C}^\times))$, for some unitary character ρ and some holomorphic map $f\colon X \to T$ to a complex torus.*

Here, a map $f\colon X \to C$ is said to be *admissible* if f is a holomorphic, surjective map to a connected, smooth complex curve C, and f has a holomorphic, surjective extension with connected fibers to smooth compactifications, $\overline{f}\colon \overline{X} \to \overline{C}$, obtained by adding divisors with normal crossings; in particular, the generic fiber of f is connected, and the induced homomorphism, $f_\sharp\colon \pi_1(X) \to \pi_1(C)$, is onto. A *complex torus* is a complex Lie group T which decomposes as a product of factors of the form $\mathbb{C}^\times$ or $S^1 \times S^1$.

For smooth, quasi-projective varieties X, the isolated points in $\mathcal{V}_1^1(X)$ are actually torsion characters, see [**10**], [**5**].

As noted in [**74**], Arapura's theorem has the following immediate corollary.

Corollary 8.3. *For a quasi-Kähler manifold X, all the components of $\mathcal{V}_d^i(X)$ passing through the origin of $\operatorname{Hom}(\pi_1(X), \mathbb{C}^\times)$ are subtori, provided one of the following conditions holds.*

(i) *$i = d = 1$.*

(ii) *X is Kähler.*

(iii) *$W_1(H^1(X, \mathbb{C})) = 0$.*

In particular, $\tau_1(\mathcal{V}_d^i(X)) = TC_1(\mathcal{V}_d^i(X))$, whenever one of the above conditions is satisfied.

In [**59**, Theorem 1.1], Libgober proves a "local" version of Arapura's theorem, in which all the translations are done by characters of finite order.

Theorem 8.4 ([**59**]). *Let $\mathcal{X}$ be a germ of a complex space with an isolated, normal singularity whose link is simply-connected, and let $\mathcal{D}$ be a divisor on $\mathcal{X}$ with n irreducible components. Then:*

(i) *The character group $\operatorname{Hom}(\pi_1(\mathcal{X} \setminus \mathcal{D}), \mathbb{C}^\times)$ is the complex algebraic torus $(\mathbb{C}^\times)^n$.*

(ii) *Each characteristic variety $\mathcal{V}_d^i(\mathcal{X} \setminus \mathcal{D})$ is a finite union of complex algebraic subtori, possibly translated by roots of unity.*

8.4. Resonance varieties. As shown in Theorem C and Corollary 7.4 of [**30**], the presence of a Kähler metric on a compact, connected, complex manifold M imposes very stringent conditions on the degree 1 resonance varieties of M. Likewise, the existence of a quasi-Kähler structure on an open manifold X puts subtle geometric constraints on $\mathcal{R}_d(X)$, provided X is 1-formal. These results—which use in an essential way Theorems 7.1 and 8.2, as well as theorems from [**23**] and [**69**] mentioned in §8.1 and §8.2—may be summarized as follows.

Theorem 8.5 ([**30**]). *Let X be a quasi-Kähler manifold, with fundamental group $G = \pi_1(X)$, and let $\{L_\alpha\}_\alpha$ be the collection of positive-dimensional, irreducible components of $\mathcal{R}_1(G)$. If G is 1-formal, then*

    (i) *Each L_α is a p-isotropic linear subspace of $H^1(G,\mathbb{C})$, of dimension at least $2p+2$, for some $p = p(\alpha) \in \{0,1\}$.*
    (ii) *If $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \{0\}$.*
    (iii) *$\mathcal{R}_d(G) = \{0\} \cup \bigcup_\alpha L_\alpha$, where the union is over all α for which $\dim L_\alpha > d + p(\alpha)$.*

Furthermore,

    (iv) *If X is a compact Kähler manifold, then G is 1-formal, and each L_α is even-dimensional and 1-isotropic.*
    (v) *If X is a smooth, quasi-projective variety, and $W_1(H^1(X,\mathbb{C})) = 0$, then G is 1-formal, and each L_α is 0-isotropic.*

Here, we say that a non-zero subspace $U \subseteq H^1(G,\mathbb{C})$ is *p-isotropic* with respect to the cup-product map $\cup_G \colon H^1(G,\mathbb{C}) \wedge H^1(G,\mathbb{C}) \to H^2(G,\mathbb{C})$ if the restriction of $\cup_G$ to $U \wedge U$ has rank p. For example, if C is a smooth complex curve with $\chi(C) < 0$, then $\mathcal{R}_1(\pi_1(C),\mathbb{C}) = H^1(C,\mathbb{C})$, and $H^1(C,\mathbb{C})$ is either 1- or 0-isotropic, according to whether C is compact or not.

8.5. Examples and applications. Theorem 8.5 can be used in a variety of ways to derive information on the fundamental groups of quasi-projective varieties—or rule out certain groups from being quasi-projective. As a first application, we obtain the following corollary, by comparing the conclusions of parts (iv) and (v).

Corollary 8.6. *Let X be a smooth, quasi-projective variety with $W_1(H^1(X,\mathbb{C})) = 0$. Let $G = \pi_1(X)$, and suppose $\mathcal{R}_1(G) \neq \{0\}$. Then G is not a Kähler group (though G is 1-formal).*

Using now Deligne's result mentioned in §8.2, we obtain a further corollary.

Corollary 8.7. *Let X be the complement of a hypersurface in $\mathbb{CP}^n$, and let $G = \pi_1(X)$. If $\mathcal{R}_1(G) \neq \{0\}$, then G is not a Kähler group.*

The assumption $\mathcal{R}_1(G) \neq \{0\}$ is really necessary. For example, take $X = \mathbb{C}^2 \setminus \{z_1 z_2 = 0\}$. Then $G = \mathbb{Z}^2$ is clearly a Kähler group, but $\mathcal{R}_1(G) = \{0\}$.

The linearity property from Theorem 8.5(i) only uses the 1-formality assumption on the group G, and follows at once from Theorem 7.1(iii). This property can be used to show that certain quasi-projective groups are not 1-formal.

Example 8.8 ([**30**]). Let $X = F(\Sigma_g, n)$ be the configuration space of n labeled points on a Riemann surface of genus g. Clearly, X is a connected, smooth, quasi-projective variety. Its fundamental group, $\pi_1(X) = P_{g,n}$, is the pure braid group on

n strings on Σ_g. The cohomology ring $H^*(F(\Sigma_g, n), \mathbb{C})$ was computed by Totaro in [89]. Using this computation, we get

$$\mathcal{R}_1(P_{1,n}) = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 0, \\ x_i y_j - x_j y_i = 0, \text{ for } 1 \le i < j < n \end{array} \right\}.$$

For $n \ge 3$, this is an irreducible, non-linear variety (a rational normal scroll). Hence, the group $P_{1,n}$ is not 1-formal.

The more refined isotropicity properties from Theorem 8.5, parts (i), (iv), and (v) use both the 1-formality and the (quasi-) Kählerianity assumptions on the group G. These isotropicity properties of the components of $\mathcal{R}_1(G)$ are utilized in [30, Theorem 11.7] to achieve a complete classification of (quasi-) Kähler groups within the class of right-angled Artin groups (which, recall, are always 1-formal).

Theorem 8.9 ([30]). *Let Γ be a finite simple graph, and G_Γ the corresponding right-angled Artin group. Then:*

(1) *G_Γ is a quasi-Kähler group if and only if Γ is a complete multipartite graph $K_{n_1,\ldots,n_r} = \overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$, in which case $G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$.*
(2) *G_Γ is a quasi-Kähler group if and only if Γ is a complete graph K_{2m}, in which case $G_\Gamma = \mathbb{Z}^{2m}$.*

8.6. Alexander polynomial. The approach we have been using so far in this section also informs on the Alexander polynomial of a (quasi-) Kähler group G. The following theorem was proved in [29], assuming G is (quasi-) projective; the same proof works in the stated generality.

Theorem 8.10 ([29]). *Let G be a quasi-Kähler group. Set $n = b_1(G)$, and let Δ_G be the Alexander polynomial of G.*

(i) *If $n \ne 2$, then the Newton polytope of Δ_G is a line segment.*
(ii) *If G is actually a Kähler group, then $\Delta_G \doteq \text{const}$.*

If $n \ge 3$, we may write $\Delta_G(t_1, \ldots, t_n) \doteq cP(t_1^{e_1} \cdots t_n^{e_n})$, for some $c \in \mathbb{Z}$, some polynomial $P \in \mathbb{Z}[t]$ equal to a product of cyclotomic polynomials, and some exponents $e_i \ge 1$ with $\gcd(e_1, \ldots, e_n) = 1$.

8.7. Bieri–Neumann–Strebel invariants. Recently, Delzant [25] found a very precise connection between the BNS invariant of a compact Kähler manifold M and admissible maps $f \colon M \to C$. (Recall that such maps, also known as *pencils*, are holomorphic, surjective maps to smooth complex curves, and have connected generic fiber.)

Theorem 8.11 (Delzant [25]). *Let M be a compact Kähler manifold, with $G = \pi_1(M)$. Then $\Sigma^1(G)^{\complement} = \bigcup f_\alpha^*\big(H^1(C_\alpha, \mathbb{R})\big)$, where the union is taken over those pencils $f_\alpha \colon M \to C_\alpha$ with the property that either $\chi(C_\alpha) < 0$, or $\chi(C_\alpha) = 0$ and f_α has some multiple fiber.*

Recall from Corollary 7.5 that the BNS invariant of a 1-formal group G is contained in the complement in $H^1(G, \mathbb{R})$ of the resonance variety $\mathcal{R}_1(G, \mathbb{R})$. In general, this inclusion is strict. Nevertheless the class of Kähler groups for which the aforementioned inclusion is an equality can be identified precisely. This is done in [74, Theorem 16.4], using the above result of Delzant, together with work of Arapura [3] and Theorem 8.5.

Theorem 8.12 ([**74**]). *Let M be a compact Kähler manifold with $b_1(M) > 0$, and let $G = \pi_1(M)$. Then $\Sigma^1(G) = \mathcal{R}_1(G, \mathbb{R})^\complement$ if and only if there is no pencil $f \colon M \to E$ onto an elliptic curve E such that f has multiple fibers.*

The equality $\Sigma^1(G) = \mathcal{R}_1(G, \mathbb{R})^\complement$ does not hold for arbitrary Kähler groups G. The following example, based on a well-known construction of Beauville [**6**], illustrates this point.

Example 8.13. Let Y be a compact Kähler manifold on which a finite group π acts freely. Let E be an elliptic curve, and let $p \colon C \to E$ be a ramified, regular π-cover, with at least one ramification point. Clearly, the diagonal action of π on $C \times Y$ is free; let $M = (C \times Y)/\pi$ be the orbit space. It is readily seen that M is a compact Kähler manifold, with $b_1(M) > 0$.

Consider the commuting diagram

(45)
$$
\begin{array}{ccc}
C \times Y & \xrightarrow{\ q\ } & M \\
{\scriptstyle \mathrm{pr}_1} \downarrow & & \downarrow {\scriptstyle f} \\
C & \xrightarrow{\ p\ } & E
\end{array}
$$

where q is the orbit map and f is the map induced by the first-coordinate projection. As noted in [**6**, Example 1.8], the fibers of f over the ramification points of p are multiple fibers, while the other (generic) fibers of f are isomorphic to Y. In other words, $f \colon M \to E$ is an elliptic pencil with multiple fibers. Let $G = \pi_1(M)$. By Theorem 8.12, then, the BNS invariant $\Sigma^1(G)$ is strictly contained in $\mathcal{R}_1(G, \mathbb{R})^\complement$.

9. Hyperplane arrangements

9.1. The complement of an arrangement. A *hyperplane arrangement* is a finite collection of hyperplanes in some complex affine space $\mathbb{C}^\ell$. The main topological object associated to an arrangement $\mathcal{A}$ is its *complement*, $X(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$. This is a smooth, quasi-projective variety, whose topological invariants are intimately connected to the combinatorics of the arrangement, as encoded in the *intersection lattice*, $L(\mathcal{A})$, which is the poset of all non-empty intersections of $\mathcal{A}$, ordered by reverse inclusion.

Example 9.1. The best-known example is the braid arrangement $\mathcal{A}_\ell$, consisting of the diagonal hyperplanes in $\mathbb{C}^\ell$. The complement is the configuration space $F(\mathbb{C}, \ell)$ of ℓ ordered points in $\mathbb{C}$, while the intersection lattice is the lattice of partitions of $[\ell] = \{1, \ldots, \ell\}$, ordered by refinement. In the early 1960s, Fox and Neuwirth showed that $\pi_1(X(\mathcal{A}_\ell)) = P_\ell$, the pure braid group on ℓ strings, while Neuwirth and Fadell showed that $X(\mathcal{A}_\ell)$ is aspherical.

For a general arrangement with complement $X = X(\mathcal{A})$, the cohomology ring $H^*(X, \mathbb{Z})$ was computed by Brieskorn in the early 1970s, building on pioneering work of Arnol'd on the cohomology ring of the braid arrangement. It follows from Brieskorn's work that the space X is formal. In 1980, Orlik and Solomon gave a simple combinatorial description of the ring $H^*(X, \mathbb{Z})$: it is the quotient $A = E/I$ of the exterior algebra E on classes dual to the meridians, modulo a certain ideal I determined by the intersection poset. We refer to the book by Orlik and Terao [**71**] for detailed explanations and further references.

The fundamental group of the complement, $G(\mathcal{A}) = \pi_1(X(\mathcal{A}))$, can be computed algorithmically, using the braid monodromy associated to a generic projection of a generic slice $\mathcal{B}$ in $\mathbb{C}^2$; see [17] and references therein. The end result is a finite presentation with generators $x_1, \ldots, x_n$ corresponding to the meridians (oriented compatibly with the complex orientations of $\mathbb{C}^2$ and the lines in $\mathcal{B}$), and commutator relators of the form $x_i \alpha_j(x_i)^{-1}$, where $\alpha_j \in P_n$ are the (pure) braid monodromy generators, acting on the meridians via the Artin representation. In particular, $H_1(G(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^n$, with preferred basis the images of the meridional generators.

It should be noted that arrangement groups are not always combinatorially determined. Indeed, G. Rybnikov has produced a pair arrangements, $\mathcal{A}$ and $\mathcal{A}'$, with $L(\mathcal{A}) \cong L(\mathcal{A}')$, but $G(\mathcal{A}) \ncong G(\mathcal{A}')$. For a detailed account of Rybnikov's celebrated example, we refer to [4].

9.2. Resonance varieties of arrangements. Let $\mathcal{A}$ be a central arrangement in $\mathbb{C}^\ell$ (i.e., all hyperplanes of $\mathcal{A}$ pass through the origin). The resonance varieties $\mathcal{R}_1^i(X(\mathcal{A}), \mathbb{C})$ were first defined and studied by Falk in [39]. The resonance varieties over an arbitrary field, $\mathcal{R}_d^i(X(\mathcal{A}), \Bbbk)$, were considered by Matei and Suciu in [64], and investigated in detail by Falk in [40]. The varieties $\mathcal{R}_d^i(X(\mathcal{A}), \Bbbk)$ lie in the affine space $\Bbbk^n$, where $n = |\mathcal{A}|$; they depend solely on the intersection lattice, $L(\mathcal{A})$, and on the characteristic of the field $\Bbbk$. A basic problem in the subject is to find concrete formulas making this dependence explicit.

Best understood are the degree 1 resonance varieties over the complex numbers, $\mathcal{R}_d(\mathcal{A}) = \mathcal{R}_d^1(X(\mathcal{A}), \mathbb{C})$. These varieties admit a very precise combinatorial description, owing to work of Falk [39], Cohen–Suciu [18], Libgober [57], Libgober–Yuzvinsky [60], and others, with the state of the art being the recent work of Falk–Yuzvinsky [41], Pereira–Yuzvinsky [75], and Yuzvinsky [91]. Let us briefly describe these varieties, based on the original approach from [39], with updates as warranted; for the latest approach, using "multinets," we refer to [41].

By the Lefschetz-type theorem of Hamm and Lê, taking a generic two-dimensional section does not change the fundamental group of the complement. Thus, in order to describe $\mathcal{R}_1(\mathcal{A}) = \mathcal{R}_1(G(\mathcal{A}))$, we may assume $\mathcal{A} = \{\ell_1, \ldots, \ell_n\}$ is an affine line arrangement in $\mathbb{C}^2$, for which no two lines are parallel. The following facts are known:

(1) The variety $\mathcal{R}_1(\mathcal{A}) \subset \mathbb{C}^n$ lies in the hyperplane $\Delta_n = \{x \in \mathbb{C}^n \mid \sum_{i=1}^n x_i = 0\}$.
(2) Each component is a linear subspace of dimension at least 2.
(3) Two distinct components of $\mathcal{R}_1(\mathcal{A})$ meet only at 0, and $\mathcal{R}_d(\mathcal{A})$ is the union of those subspaces of dimension greater than d.

Example 9.2. Let $\mathcal{A}$ be a pencil of n lines through the origin of $\mathbb{C}^2$, defined by the equation $z_1^n - z_2^n = 0$. The fundamental group of the complement is $G = \langle x_1, \ldots, x_n \mid x_1 \cdots x_n \text{ central} \rangle$. If $n = 1$ or 2, then $\mathcal{R}_1(\mathcal{A}) = \{0\}$; otherwise, $\mathcal{R}_1(\mathcal{A}) = \cdots = \mathcal{R}_{n-2}(\mathcal{A}) = \Delta_n$, and $\mathcal{R}_{n-1}(\mathcal{A}) = \{0\}$.

Returning to the general case, the simplest components of $\mathcal{R}_1(\mathcal{A})$ are the *local* components: to an intersection point $v_J = \bigcap_{j \in J} \ell_j$ of multiplicity $|J| \geq 3$, there corresponds a subspace L_J of dimension $|J| - 1$, given by equations of the form $\sum_{j \in J} x_j = 0$, and $x_i = 0$ if $i \notin J$.

The remaining components correspond to certain "neighborly partitions" of sub-arrangements of $\mathcal{A}$: to each such partition Π, there corresponds a subspace L_Π,

given by equations of the form $\sum_{j\in\pi} x_j = 0$, with π running through the blocks of Π. Moreover, $\dim L_\Pi > 0$ if and only if a certain bilinear form associated to Π is degenerate.

If $|\mathcal{A}| \leq 5$, then all components of $\mathcal{R}_1(\mathcal{A})$ are local. For $|\mathcal{A}| \geq 6$, though, the resonance variety $\mathcal{R}_1(\mathcal{A})$ may have interesting components.

Example 9.3. Let $\mathcal{A}$ be a generic 3-slice of the braid arrangement $\mathcal{A}_4$, with defining polynomial $Q(\mathcal{A}) = z_0 z_1 z_2 (z_0 - z_1)(z_0 - z_2)(z_1 - z_2)$. Take a generic plane section, and label the corresponding lines as $6, 2, 4, 3, 5, 1$. The variety $\mathcal{R}_1(\mathcal{A}) \subset \mathbb{C}^6$ has 4 local components, corresponding to the triple points $124, 135, 236, 456$, and one non-local component, corresponding to the neighborly partition $\Pi = (16|25|34)$:

$$L_{124} = \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\}, \ L_{135} = \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\},$$

$$L_{236} = \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\}, \ L_{456} = \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\},$$

$$L_\Pi = \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}.$$

Since all these components are 2-dimensional, $\mathcal{R}_2(\mathcal{A}) = \{0\}$.

For an arbitrary arrangement $\mathcal{A}$, it follows from the work of Falk, Pereira, and Yuzvinsky [**41, 75, 91**] that any non-local component in $\mathcal{R}_1(\mathcal{A})$ has dimension either 2 or 3. The only known example for which non-local components of dimension 3 occur is the Hessian arrangement of 12 planes in $\mathbb{C}^3$, defined by the polynomial $Q(\mathcal{A}) = z_0 z_1 z_2 \prod_{j=0}^{2} \prod_{k=0}^{2} (z_0 + \omega^j z_1 + \omega^k z_2)$, where $\omega = \exp(2\pi i/3)$.

9.3. Characteristic varieties of arrangements.
Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a central arrangement in $\mathbb{C}^\ell$, and let $G(\mathcal{A}) = \pi_1(X(\mathcal{A}))$ be the fundamental group of its complement. From the discussion in §9.1, we know that $G(\mathcal{A})$ is a quasi-projective, 1-formal group. Let $\mathcal{V}_d(\mathcal{A}) = \mathcal{V}_d(G(\mathcal{A}), \mathbb{C})$ be its (degree 1) characteristic varieties.

From Arapura's Theorem 8.2, we know that $\mathcal{V}_d(\mathcal{A})$ consists of subtori in $(\mathbb{C}^\times)^n$, possibly translated by roots of unity (necessarily of finite order, if $d = 1$), together with a finite number of isolated unitary characters. By Theorem 7.1, we have

$$(46) \qquad\qquad TC_1(\mathcal{V}_d(\mathcal{A})) = \mathcal{R}_d(\mathcal{A}), \quad \text{for all } d \geq 1.$$

The tangent cone formula (46) for complements of hyperplane arrangements was first proved (by different methods) by Cohen and Suciu [**18**], Libgober [**58**], and Libgober and Yuzvinsky [**60**], and was generalized to the higher-degree jump loci by Cohen and Orlik [**15**].

As an upshot of (46), the components of $\mathcal{V}_1(\mathcal{A})$ passing through the origin are completely determined by $\mathcal{R}_1(\mathcal{A})$, and thus, by the intersection lattice, $L(\mathcal{A})$: to each (linear) component $L \subset \mathbb{C}^n$ of $\mathcal{R}_1(\mathcal{A})$ there corresponds a (torus) component $T = \exp(L) \subset (\mathbb{C}^\times)^n$ of $\mathcal{V}_1(\mathcal{A})$. The only novelty here is that some of these subtori may intersect away from the origin, sometimes in a character belonging to $\mathcal{V}_2(\mathcal{A})$—a phenomenon first noted in [**18**, Example 4.4].

Example 9.4. For the arrangement $\mathcal{A}$ from Example 9.3, the variety $\mathcal{V}_1(\mathcal{A})$ has four local components—$V_{124}, V_{135}, V_{236},$ and V_{456}—and one non-local component, V_Π. Any two of these components intersect only at 1; moreover, $\mathcal{V}_2(\mathcal{A}) = \{1\}$. Computing the Alexander matrix of $G(\mathcal{A})$ and applying Proposition 4.9 reveals that there are no translated components in $\mathcal{V}_1(\mathcal{A})$.

Not too much is known about the components of $\mathcal{V}_1(\mathcal{A})$ not passing through 1. A major open problem in the subject is to decide whether such "translated" components are combinatorially determined. That this is not a moot point is illustrated by the following example from [84].

Example 9.5. Let $\mathcal{A}$ be the deleted B_3 arrangement, with defining polynomial $Q(\mathcal{A}) = z_0 z_1 (z_0^2 - z_1^2)(z_0^2 - z_2^2)(z_1^2 - z_2^2)$. The variety $\mathcal{V}_1(\mathcal{A})$ contains 7 local components, corresponding to 6 triple points and one quadruple point, and 5 non-local components passing through 1, corresponding to braid sub-arrangements. Additionally, $\mathcal{V}_1(\mathcal{A})$ contains a component of the form $\rho \cdot T$, where

$$T = \{(t^2, t^{-2}, 1, 1, t^{-1}, t^{-1}, t, t) \mid t \in \mathbb{C}^{\times}\}$$

is a 1-dimensional subtorus of $(\mathbb{C}^{\times})^8$, and $\rho = (1, 1, -1, -1, -1, -1, 1, 1)$ is a root of unity of order 2.

In [28], Dimca provides an effective algorithm for detecting translated tori in the characteristic varieties of arrangements, such as the component $\rho \cdot T$ arising in the above example. In [70], Nazir and Raza prove the following result: If $\mathcal{A}$ has 2 lines (or less) that contain all intersection points of multiplicity 3 and higher, then $\mathcal{V}_1(\mathcal{A})$ has no translated components. For the deleted B_3 arrangement, there exist precisely 3 lines containing all high-multiplicity points, and so Nazir and Raza's result is best possible from this point of view.

9.4. Arrangements with parallel lines. The theory developed so far works with only minimal modifications for arbitrary affine line arrangements in $\mathbb{C}^2$, i.e, for arrangements $\mathcal{A} = \{\ell_1, \ldots, \ell_n\}$ which may contain parallel lines. Homogenizing, we get an arrangement $\bar{\mathcal{A}} = \{L_0, L_1, \ldots, L_n\}$, with $L_0 = \mathbb{CP}^2 \setminus \mathbb{C}^2$ the line at infinity. Viewing $\bar{\mathcal{A}}$ as the projectivization of a central arrangement $\hat{\mathcal{A}}$ in $\mathbb{C}^3$, and taking a generic 2-section $\mathcal{B}$ of $\hat{\mathcal{A}}$, we are back to the situation studied previously.

Remark 9.6. Note that the complement of $\mathcal{A}$ in $\mathbb{C}^2$ is the same as the complement of $\bar{\mathcal{A}}$ in $\mathbb{CP}^2$. Thus, the arrangement group, $G(\mathcal{A}) = \pi_1(X(\mathcal{A}))$, can be viewed as the fundamental group of the complement of a projective hypersurface.

It is now an easy matter to relate the jump loci of $\mathcal{A}$ to those of $\mathcal{B}$. For instance, the resonance variety $\mathcal{R}_1(\mathcal{A}) \subset \mathbb{C}^n$ may be obtained from $\mathcal{R}_1(\mathcal{B}) \subset \mathbb{C}^{n+1}$ by slicing with a suitable hyperplane.

Example 9.7. Let $\mathcal{A}$ be an arrangement of n parallel lines in $\mathbb{C}^2$. Clearly, $X(\mathcal{A}) = (\mathbb{C} \setminus \{n \text{ points}\}) \times \mathbb{C}$, and thus $G(\mathcal{A}) = F_n$. It is readily seen that $\mathcal{B}$ is a pencil of $n+1$ lines in $\mathbb{C}^2$, and thus, $G(\mathcal{B}) \cong \mathbb{Z} \times F_n$. This isomorphism identifies $\mathcal{R}_1(\mathcal{B}) = \Delta_n$ with $\mathcal{R}_1(\mathcal{A}) = \mathbb{C}^n$.

In general, to a family of $k \geq 2$ parallel lines in $\mathcal{A}$ there corresponds a pencil of $k+1$ lines in $\mathcal{B}$. The corresponding k-dimensional local component of $\mathcal{R}_1(\mathcal{B})$ yields a k-dimensional component of $\mathcal{R}_1(\mathcal{A})$. Similar considerations apply to non-local components.

Example 9.8. Let $\mathcal{A}$ be the line arrangement defined by the polynomial $Q(\mathcal{A}) = z_1 z_2 (z_1 - 1)(z_2 - 1)(z_1 - z_2)$. The corresponding central arrangement, $\hat{\mathcal{A}}$, is the arrangement from Example 9.3. Thus, $\mathcal{R}_1(\mathcal{A})$ has a (unique) non-local component, $L = \{x_1 + x_2 + x_3 = x_2 - x_5 = x_3 - x_4 = 0\}$.

9.5. Zariski-type theorems. A well-known theorem of Zariski asserts that an arrangement of lines in $\mathbb{CP}^2$ has only double points if and only if the fundamental group of its complement is free abelian; see [**12**, Theorem 1.1] for details and references. One may wonder whether there are other classes of line arrangements which may be similarly characterized in terms of their groups.

In [**12**, Corollary 1.7], Choudary, Dimca, and Papadima prove an analogue of Zariski's theorem, in the setting of affine arrangements. Given an r-tuple of integers $m = (m_1, \ldots, m_r)$, a line arrangement $\mathcal{A}$ in $\mathbb{C}^2$ is said to be of type $\mathcal{A}(m)$ if $\mathcal{A}$ consists of lines in r parallel directions, each direction containing m_i lines. Clearly, such arrangements have only double points, and any nodal arrangement in $\mathbb{C}^2$ is of type $\mathcal{A}(m)$, for some r-tuple m.

Theorem 9.9 ([**12**])**.** *Let $\mathcal{A}$ be a line arrangement in $\mathbb{C}^2$, with group $G = G(\mathcal{A})$. The following are equivalent:*

    (i) *$\mathcal{A}$ is of type $\mathcal{A}(m)$, for some r-tuple $m = (m_1, \ldots, m_r)$.*

    (ii) *G is isomorphic to $F_{m_1} \times \cdots \times F_{m_r}$, via an isomorphism preserving the standard generators in H_1.*

In particular, if a line arrangement $\mathcal{A}$ has only double points, then the fundamental group of its complement is isomorphic to a finite direct product of finitely generated free groups.

The next result was inspired by recent work of Fan [**43**]. The implication (i) $\Rightarrow$ (iv) below recovers the main result (Theorem 1) from [**43**]. It also improves on Theorem 9.9 in the particular case when $r = 1$, by allowing arbitrary isomorphisms between G and a free group. We offer a completely different—and much shorter— proof of this implication, based on the techniques described so far.

Theorem 9.10. *Let $\mathcal{A} = \{\ell_1, \ldots, \ell_n\}$ be an arrangement of lines in $\mathbb{C}^2$, with group $G = G(\mathcal{A})$. The following are equivalent:*

    (i) *The group G is a free group.*

    (ii) *The characteristic variety $\mathcal{V}_1(\mathcal{A})$ coincides with $(\mathbb{C}^\times)^n$.*

    (iii) *The resonance variety $\mathcal{R}_1(\mathcal{A})$ coincides with $\mathbb{C}^n$.*

    (iv) *The lines $\ell_1, \ldots, \ell_n$ are all parallel.*

PROOF. (i) $\Rightarrow$ (ii). If G is a free group (necessarily, of rank n), then, as noted in Example 2.3, $\mathcal{V}_1(G) = (\mathbb{C}^\times)^n$.

(ii) $\Rightarrow$ (iii). If $\mathcal{V}_1(G) = (\mathbb{C}^\times)^n$, then obviously $TC_1(\mathcal{V}_1(G)) = \mathbb{C}^n$. Hence, by formula (33) (or, alternatively, formula (46)), $\mathcal{R}_1(G) = \mathbb{C}^n$.

(iii) $\Rightarrow$ (iv). Suppose $\mathcal{A}$ has a point of multiplicity $k \geq 3$. Then $\mathcal{R}_1(\mathcal{A})$ has a (local) component of dimension $k - 1$. But $k \leq n$, and so this contradicts $\mathcal{R}_1(\mathcal{A}) = \mathbb{C}^n$. Thus, all multiple points of $\mathcal{A}$ must be double points, i.e, $\mathcal{A}$ must be of type $\mathcal{A}(m)$, for some r-tuple $m = (m_1, \ldots, m_r)$. To each family of $m_i \geq 2$ parallel lines, there corresponds a component of $\mathcal{R}_1(\mathcal{A})$ of dimension m_i. But $m_i < n$, unless $r = 1$ and $m_1 = n$, i.e., $\mathcal{A}$ consists of n parallel lines.

(iv) $\Rightarrow$ (i). If the lines are parallel, then, as noted in Example 9.7, $G = F_n$. $\quad\square$

9.6. Comparison with Kähler groups. Recall that arrangement groups are 1-formal, quasi-projective groups. But are they Kähler groups? Of course, a necessary condition for $G = G(\mathcal{A})$ to be a Kähler group is that $b_1(G) = |\mathcal{A}|$ must be even. In [**2**, Example 3.48], it is noted that the group of a pencil of $n \geq 3$ lines cannot be realized as a "non-fibered" Kähler group, i.e., the fundamental group of

a compact Kähler manifold which does not fiber over a Riemann surface of genus $g \geq 2$.

In the next theorem, we go much further, and identify precisely which arrangement groups are Kähler groups.

Theorem 9.11. *Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{C}^2$, with group $G = G(\mathcal{A})$. The following are equivalent:*

  (i) *G is a Kähler group.*
  (ii) *G is a free abelian group of even rank.*
  (iii) *$\mathcal{A}$ is an arrangement of an even number of lines in general position.*

PROOF. Implication (iii) $\Rightarrow$ (ii) follows from Zariski's theorem, while (ii) $\Rightarrow$ (i) follows from the discussion in §8.1.

To prove (i) $\Rightarrow$ (iii), assume $G = G(\mathcal{A})$ is a Kähler group. If $\mathcal{A}$ is not in general position, i.e, $\mathcal{A}$ is not of type $\mathcal{A}(1, \ldots, 1)$, then either $\mathcal{A}$ has an intersection point of multiplicity $k + 1 \geq 3$, or $\mathcal{A}$ contains a family of $k \geq 2$ parallel lines. In either case, $\mathcal{R}_1(\mathcal{A})$ has a k-dimensional component; in particular, $\mathcal{R}_1(\mathcal{A}) \neq \{0\}$. In view of Remark 9.6, the conclusion follows from Corollary 8.7. $\square$

9.7. Comparison with right-angled Artin groups. Arrangement groups also share many common features with right-angled Artin groups. For instance, if G is a group in either class, then G is a commutator-relators group; G is 1-formal; and each resonance variety $\mathcal{R}_d(G)$ is a union of linear subspaces. Moreover, the free groups F_n and the free abelian groups $\mathbb{Z}^n$ belong to both classes. So what exactly is the intersection of these two classes of naturally defined groups?

To answer this question, we need one more definition, due to Fan [**42**]. Given a line arrangement $\mathcal{A}$, its *multiplicity graph*, $\Gamma(\mathcal{A})$, is the graph with vertices the intersection points with multiplicity at least 3, and edges the segments between multiple points on lines which pass through more than one multiple point.

Theorem 9.12. *Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{C}^2$, with group $G = G(\mathcal{A})$. The following are equivalent:*

  (i) *G is a right-angled Artin group.*
  (ii) *G is a finite direct product of finitely generated free groups.*
  (iii) *The multiplicity graph $\Gamma(\mathcal{A})$ is a forest.*

PROOF. Recalling that every arrangement group is a quasi-Kähler group, equivalence (i) $\Leftrightarrow$ (ii) follows at once from Theorem 8.9(1).

Implication (iii) $\Rightarrow$ (ii) is proved in [**42**, Theorem 1], while (ii) $\Rightarrow$ (iii) is proved in [**37**, Theorem 6.1], if we keep in mind Remark 2.8 from that paper. $\square$

9.8. Alexander polynomials of arrangements. As we saw in Theorem 4.10, there is a strong relationship between the Alexander polynomial of a group G and the codimension-1 strata of its degree-1 jump loci. This relationship was used in [**29**, Proposition 3.4] to compute the Alexander polynomial of $G = \pi_1(X(\mathcal{A}))$, in the case when $\mathcal{A}$ is a generic plane section of a central hyperplane arrangement. A similar proof works for arbitrary affine line arrangements. For the sake of completeness, we go over the argument, with the required modifications.

Let $\mathcal{A}$ be an arrangement of n lines in $\mathbb{C}^2$, with group $G = G(\mathcal{A})$. Recall $H = \mathrm{ab}(G)$ is isomorphic to $\mathbb{Z}^n$, and comes equipped with a preferred basis, corresponding to the oriented meridians around the lines; this yields an identification of

$\mathbb{Z}H$ with $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. Define the Alexander polynomial of the arrangement to be $\Delta_{\mathcal{A}} := \Delta_G$, viewed as a Laurent polynomial in Λ. This polynomial depends (up to normalization) only on the group G and its peripheral structure; thus, only on the homeomorphism type of the complement $X(\mathcal{A})$.

Example 9.13. Let $\mathcal{A}$ a pencil of n lines. Recall we have $G = \langle x_1, \ldots, x_n \mid x_1 \cdots x_n \text{ central}\rangle$. It is readily checked that $\Delta_{\mathcal{A}} = 0$ if $n = 1$; $\Delta_{\mathcal{A}} = 1$ if $n = 2$; and $\Delta_{\mathcal{A}} = (t_1 \cdots t_n - 1)^{n-2}$, otherwise.

Example 9.14. Let $\mathcal{A}$ be an arrangement of $n - 1$ parallel lines, together with a transverse line, i.e., an arrangement of type $\mathcal{A}(n - 1, 1)$. Then $G = \langle x_1, \ldots, x_n \mid x_n \text{ central}\rangle = F_{n-1} \times \mathbb{Z}$. It is readily checked that $\Delta_{\mathcal{A}} = 0$ if $n = 1$; $\Delta_{\mathcal{A}} = 1$ if $n = 2$; and $\Delta_{\mathcal{A}} = (t_n - 1)^{n-2}$, otherwise.

Note that, for $n \geq 3$, the corresponding pair of arrangements have isomorphic groups, but non-homeomorphic complements: the difference is picked up by the Alexander polynomial.

Theorem 9.15. *Let $\mathcal{A}$ be an arrangement of n lines in $\mathbb{C}^2$, with Alexander polynomial $\Delta_{\mathcal{A}}$.*

  (i) *If $\mathcal{A}$ is a pencil and $n \geq 3$, then $\Delta_{\mathcal{A}} = (t_1 \cdots t_n - 1)^{n-2}$.*
  (ii) *If $\mathcal{A}$ is of type $\mathcal{A}(n - 1, 1)$ and $n \geq 3$, then $\Delta_{\mathcal{A}} = (t_n - 1)^{n-2}$.*
  (iii) *For all other arrangements, $\Delta_{\mathcal{A}} \doteq \text{const}$.*

PROOF. Cases (i) and (ii) have been treated already in the previous two examples. Thus, we may assume $\mathcal{A}$ does not belong to either class; in particular, $n \geq 3$.

From §§9.3–9.4, we know that intersection points of multiplicity $k + 1 \geq 3$, and families of $k \geq 2$ parallel lines give rise to local components of $\mathcal{V}_1(\mathcal{A})$, of dimension k. In both situations, we must have $k \leq n - 2$, since we are in case (iii).

If $n \leq 5$, then all components of $\mathcal{V}_1(\mathcal{A})$ are local, except if $\mathcal{A}$ is the arrangement of 5 lines from Example 9.8, in which case $\mathcal{V}_1(\mathcal{A})$ has a 2-dimensional global component. If $n \geq 6$, the variety $\mathcal{V}_1(\mathcal{A})$ may very well have non-local components; yet, as shown in [75, Theorem 7.2], the dimension of any such component is at most 4.

Putting things together, we see that all components of $\mathcal{V}_1(\mathcal{A})$ must have codimension at least 2; in other words, the codimension-1 stratum of this variety, $\check{\mathcal{V}}_1(\mathcal{A})$, is empty. Since $n \geq 2$, Theorem 4.10(iii) implies that $\Delta_{\mathcal{A}} \doteq \text{const}$. □

9.9. Σ-invariants of arrangements. Let $\mathcal{A}$ be a central arrangement in $\mathbb{C}^\ell$, with complement $X = X(\mathcal{A})$. It follows from the work of Esnault, Schechtman, and Viehweg [38] that the exponential map, $\exp \colon H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^\times)$, induces an isomorphism of analytic germs, $\exp \colon (\mathcal{R}_d^i(X, \mathbb{C}), 0) \xrightarrow{\simeq} (\mathcal{V}_d^i(X, \mathbb{C}), 1)$, for all $i \geq 0$ and all $d > 0$. In particular, the resonance varieties $\mathcal{R}_d^i(X, \mathbb{C})$ are all finite unions of rationally defined linear subspaces. Formula (41) now yields the following combinatorial upper bound for the BNSR invariants of an arrangement.

Proposition 9.16 ([74]). *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$, with complement X. Then $\Sigma^q(X, \mathbb{Z}) \subseteq \left(\bigcup_{i \leq q} \mathcal{R}_1^i(X, \mathbb{R}) \right)^{\complement}$, for all $q \geq 0$.*

On the other hand, it is known from the work of Eisenbud, Popescu, and Yuzvinsky [36] that resonance "propagates" for the Orlik-Solomon algebra of an arrangement. More precisely, if $i < j \leq \ell$, then $\mathcal{R}_1^i(X, \mathbb{C}) \subseteq \mathcal{R}_1^j(X, \mathbb{C})$. Using this fact, we get the following immediate corollary.

Corollary 9.17. *Let X be a hyperplane arrangement complement. Then $\Sigma^q(X, \mathbb{Z}) \subseteq \mathcal{R}_1^q(X, \mathbb{R})^{\complement}$, for all $q \geq 0$.*

The obvious question now is whether equality holds in the above corollary. In the case when $q = 1$, this is a question about the BNS invariant of the arrangement group. We state this question (as well as a couple of related ones), as follows.

Question 9.18. Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{C}^2$, with group $G = G(\mathcal{A})$.

   (i) Does the equality $\Sigma^1(G) = -\Sigma^1(G)$ hold?
   (ii) Even stronger, does the equality $\Sigma^1(G) = \mathcal{R}_1(G, \mathbb{R})^{\complement}$ hold?
   (iii) If it doesn't, is the BNS invariant combinatorially determined?

For a complexified real arrangement, complex conjugation in $\mathbb{C}^2$ restricts to a homeomorphism $\alpha\colon X \to X$ with the property that $\alpha_* = -\operatorname{id}$ on $H_1(X, \mathbb{Z})$. It follows that $\Sigma^1(G) = -\Sigma^1(G)$, which is consistent with the symmetry property of $\mathcal{R}_1(G, \mathbb{R})^{\complement}$.

10. Milnor fibrations

10.1. The Milnor fibration of a polynomial. Let $f \in \mathbb{C}[z_0, \ldots, z_d]$ be a weighted homogeneous polynomial of degree n, with positive integer weights $(w_0, \ldots, w_d)$ assigned to the variables. Denote by $V = V(f)$ the zero-set of the polynomial function $f\colon \mathbb{C}^{d+1} \to \mathbb{C}$, and let $X := \mathbb{C}^{d+1} \setminus V$ be the complement of this variety.

As shown by Milnor [68], the restriction $f\colon X \to \mathbb{C}^\times$ is the projection map of a smooth, locally trivial bundle, nowadays known as the (global) *Milnor fibration* of f. The *Milnor fiber*, $F = F(f)$, is simply the typical fiber of this fibration, $F := f^{-1}(1)$. The Milnor fiber of f is a smooth affine variety, having the homotopy type of a d-dimensional, finite CW-complex. We will assume throughout that f is not a proper power, so that the Milnor fiber F is connected.
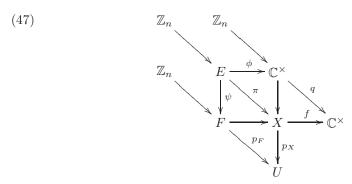
The *geometric monodromy* of the Milnor fibration is the map $h\colon F \to F$, $(z_0, \ldots, z_d) \mapsto (\zeta_n^{w_0} z_0, \ldots, \zeta_n^{w_d} z_d)$, where $\zeta_n = \exp(2\pi i/n)$. Plainly, the mapping torus of h is homeomorphic to the complement X. Moreover, the map h generates a cyclic group $\mathbb{Z}_n$ which acts freely on F. This free action gives rise to a regular n-fold covering, $F \to F/\mathbb{Z}_n$.

10.2. The Milnor fiber as a finite cyclic cover. Let us describe the cover $F \to F/\mathbb{Z}_n$ more concretely, at least in the case when f is homogeneous. Following the approach from [16] (with a few more details added in for completeness), we build an interlocking set of fibrations in which this cover fits. These fibrations will be recorded in Diagram (47) below.

Start with the map $p\colon \mathbb{C}^{d+1} \setminus \{0\} \to \mathbb{CP}^d$, $z = (z_0, \ldots, z_d) \mapsto (z_0 : \cdots : z_d)$; this is the projection map of the Hopf bundle, with fiber $\mathbb{C}^\times$. Denote by U the complement of the hypersurface in $\mathbb{CP}^d$ defined by f. The map p restricts to a bundle map, $p_X\colon X \to U$, also with fiber $\mathbb{C}^\times$. By homogeneity, $f(wz) = w^n f(z)$, for every $z \in X$ and $w \in \mathbb{C}^\times$. Thus, the restriction of f to a fiber of p_X may be identified with the covering projection $q\colon \mathbb{C}^\times \to \mathbb{C}^\times$, $q(w) = w^n$.

Let $E = \{(z, w) \in X \times \mathbb{C}^\times \mid f(z) = w^n\}$, and let $\pi\colon E \to X$ and $\phi\colon E \to \mathbb{C}^\times$ be the restrictions of the coordinate projections on $X \times \mathbb{C}^\times$. Clearly, $f \circ \pi = q \circ \phi$.

Thus, π is a regular $\mathbb{Z}_n$-cover, obtained as the pullback of q along f.

(47)

$$
\begin{array}{ccc}
\mathbb{Z}_n & & \mathbb{Z}_n \\
\mathbb{Z}_n & E \xrightarrow{\phi} \mathbb{C}^\times & \\
& \psi \downarrow \quad \pi \quad \downarrow \quad q & \\
& F \longrightarrow X \xrightarrow{f} \mathbb{C}^\times \\
& p_F \searrow \quad \downarrow p_X & \\
& U &
\end{array}
$$

Now let $p_F \colon F \to U$ be the restriction of p_X to the Milnor fiber. It is readily seen that this is the orbit map of the free action of the geometric monodromy on F; hence, $F/\mathbb{Z}_n \cong U$.

Finally, given an element $(z, w) \in E$, note that $f(w^{-1}z) = w^{-n}f(z) = 1$. Hence, we may define a map $\psi \colon E \to F$ by $\psi(z, w) = w^{-1}z$. Plainly, $p_F \circ \psi = p_X \circ \pi$. Therefore, the pullback of p_F along p_X coincides with the covering map $\pi \colon E \to X$.

10.3. Identifying the cyclic cover. We now determine the classifying homomorphism $\lambda \colon \pi_1(U) \twoheadrightarrow \mathbb{Z}_n$ of the regular, cyclic n-fold cover $p_F \colon F \to U$.

Let $f = f_1^{a_1} \cdots f_s^{a_s}$ be the factorization into distinct irreducible factors of f. Recall we assume f is not a proper power, i.e., $\gcd(a_1, \ldots, a_s) = 1$. The hypersurface V decomposes into irreducible components, $V_i = f_i^{-1}(0)$. Choose as generators of $\pi_1(X)$ meridian circles γ_i about V_i, with orientations compatible with the complex orientations of $\mathbb{C}^{d+1}$ and V_i.

The map $f_\sharp \colon \pi_1(X) \to \mathbb{Z}$ factors through $f_* \colon H_1(X, \mathbb{Z}) \to \mathbb{Z}$, which is given by $f_*([\gamma_i]) = a_i$, see [**27**, pp. 76–77]. On the other hand, the map $(p_X)_* \colon H_1(X, \mathbb{Z}) \to H_1(U, \mathbb{Z})$ may be identified with the canonical projection $\mathbb{Z}^s \twoheadrightarrow \mathbb{Z}^s/(n_1, \ldots, n_s)$, where $n_i = \deg(f_i)$, see [**27**, pp. 102–103].

Putting these observations together, we obtain the following generalization of Proposition 1.2 from [**16**].

Proposition 10.1. *The classifying homomorphism of the covering $p_F \colon F \to U$ is the composition $\lambda = \alpha \circ \mathrm{ab}$, where $\mathrm{ab} \colon \pi_1(U) \to H_1(U) \cong \mathbb{Z}^s/(n_1, \ldots, n_s)$ is the abelianization map, and $\alpha \colon H_1(U) \to \mathbb{Z}_n$ is induced by the composition $\mathbb{Z}^s \xrightarrow{(a_1, \ldots, a_s)} \mathbb{Z} \twoheadrightarrow \mathbb{Z}_n$.*

Alternatively, if we identify $\pi_1(U) = \pi_1(X)/\langle \gamma_1^{n_1} \cdots \gamma_s^{n_s} \rangle$ and write $\mathbb{Z}_n = \langle g \mid g^n = 1 \rangle$, then the epimorphism $\lambda \colon \pi_1(U) \twoheadrightarrow \mathbb{Z}_n$ is given by $\lambda(\gamma_i) = g^{a_i}$.

10.4. Homology of the Milnor fiber. In [**68**], Milnor determined the homotopy type of the Milnor fiber in an important special case: If $f = f(z_0, \ldots, z_d)$ has an isolated critical point at 0 of multiplicity μ, then $F(f) \simeq \bigvee^\mu S^d$. As shown by Milnor and Orlik, the multiplicity can be calculated in terms of the degree and the weights as $\mu = (n/w_0 - 1) \cdots (n/w_d - 1)$.

For non-isolated singularities, though, no general formula for the homology of the Milnor fiber is known. Nevertheless, Theorem 3.1, together with Proposition

10.1, yields a formula for the homology groups $H_1(F, \Bbbk)$, in terms of the characteristic varieties $V_d(U, \Bbbk)$ of the projectivized complement, under suitable assumptions on the field $\Bbbk$.

Theorem 10.2. *Let $f = f_1^{a_1} \cdots f_s^{a_s}$ be a homogeneous polynomial of degree n, with irreducible factors f_i, and with $\gcd(a_1, \ldots, a_s) = 1$. Let $\Bbbk$ be an algebraically closed field of characteristic not dividing n, and fix a primitive n-th root of unity $\zeta \in \Bbbk$. Then,*

$$
(48) \qquad \dim_{\Bbbk} H_1(F, \Bbbk) = s - 1 + \sum_{1 \neq k \mid n} \varphi(k) \operatorname{depth}_{\Bbbk} \left(\rho^{n/k} \right),
$$

where $\rho \colon \pi_1(U) \to \Bbbk^{\times}$ is the character defined on meridians by $\rho(\gamma_i) = \zeta^{a_i}$.

In the case where $\deg(f_i) = 1$ for $1 \leq i \leq s$, this result reduces to Theorem 4.4 from [14].

10.5. Milnor fibers of arrangements. Much effort has been put over the past two decades in computing the homology groups of the Milnor fiber of a hyperplane arrangement. For our purposes here, we shall restrict to central arrangements in $\mathbb{C}^3$, defined by polynomials of the form $f = f_1 \cdots f_n$, with $f_i = f_i(z_0, z_1, z_2)$ linear forms.

Let $\mathcal{A}$ be such an arrangement, with complement X, projectivized complement U, and meridians $\gamma_1, \ldots, \gamma_n$. Since $\mathbb{CP}^d \setminus \{f_i = 0\} \cong \mathbb{C}^d$, the bundle $p_X \colon X \to U$ is trivial. This gives a diffeomorphism $X \cong U \times \mathbb{C}^{\times}$, under which the group $\pi_1(\mathbb{C}^{\times})$ corresponds to $\mathbb{Z} = \langle \gamma_1 \cdots \gamma_n \rangle$. Thus, the characteristic varieties of $\pi_1(U)$ are given by

$$
(49) \qquad \mathcal{V}_d(U, \Bbbk) = \{ t \in (\Bbbk^{\times})^n \mid t \in \mathcal{V}_d(X, \Bbbk) \text{ and } t_1 \cdots t_n = 1 \}.
$$

In particular, $\mathcal{V}_d(U, \mathbb{C}) \subset (\mathbb{C}^{\times})^n$ is the intersection of $\mathcal{V}_d(\mathcal{A}) = \mathcal{V}_d(X, \mathbb{C})$ with the subtorus $(\mathbb{C}^{\times})^{n-1} = \{ t_1 \cdots t_n = 1 \}$.

Some well-known questions arise.

Question 10.3. Let $F = \{ f = 1 \}$ be the Milnor fiber of a central arrangement $\mathcal{A}$ in $\mathbb{C}^3$.

    (i) Is $H_1(F, \mathbb{Z})$ torsion-free?
    (ii) Is $H_1(F, \mathbb{Z})$ determined by the intersection lattice $L(\mathcal{A})$?
    (iii) Is $b_1(F)$ determined by $L(\mathcal{A})$?

Of course, an affirmative answer to (ii) implies one for (iii), while an affirmative answer to both (i) and (iii) implies one for (ii). In view of Theorem 10.2, if one could show that all translated components of $\mathcal{V}_d(U, \mathbb{C})$ are combinatorially determined, one would obtain an affirmative answer to (iii).

Example 10.4. Let $\mathcal{A}$ be a generic 3-slice of the braid arrangement, with defining polynomial $f = z_0 z_1 z_2 (z_0 - z_1)(z_0 - z_2)(z_1 - z_2)$. Let U be the projectivized complement of $\mathcal{A}$, and let $\rho \colon \pi_1(U) \to \mathbb{C}, \gamma_i \mapsto \zeta_6$ be the character corresponding to the Milnor fiber F. Recalling from Example 9.4 the decomposition into irreducible components of $\mathcal{V}_1(U)$, we see that λ^2 belongs to V_{Π}, yet $\lambda \notin \mathcal{V}_1(U)$. Since $\mathcal{V}_2(U) = \{ 1 \}$, we find

$$
b_1(F) = 5 + \varphi(3) \cdot \operatorname{depth}_{\mathbb{C}}(\lambda^2) = 5 + 2 \cdot 1 = 7.
$$

Direct computation with the Jacobian matrix of $\pi_1(U)$, based on the method outlined in Proposition 4.1 shows that, in fact, $H_1(F, \mathbb{Z}) = \mathbb{Z}^7$.

If one is willing to go to multi-arrangements, then the answer to Question 10.3(i) is negative, as the following example from [14, §7.3] illustrates.

Example 10.5. Let $\mathcal{A}$ be the be the deleted B_3 arrangement discussed in Example 9.5. Assigning weights $(2, 1, 3, 3, 2, 2, 1, 1)$ to the factors of $Q(\mathcal{A})$, we obtain a new homogeneous polynomial (of degree 15),

$$f = z_0^2 z_1 (z_0^2 - z_1^2)^3 (z_0^2 - z_2^2)^2 (z_1^2 - z_2^2).$$

Note that $\mathcal{A}$ and $V(f)$ share the same (projectivized) complement, yet the Milnor fibers of $Q(\mathcal{A})$ and f are different. Applying Theorem 10.2 to the Milnor fiber $F = F(f)$, we find that $\dim_{\Bbbk} H_1(F, \Bbbk) = 7$ if char $\Bbbk \neq 2, 3$, or 5, yet $\dim_{\Bbbk} H_1(F, \Bbbk) = 9$ if char $\Bbbk = 2$.

Direct computation with the Jacobian matrix of $\pi_1(U)$ yields the precise formula for the homology with integer coefficients:

$$H_1(F, \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

10.6. Formality of the Milnor fiber. The following question was raised in [73]: Is the Milnor fiber of a reduced polynomial, $F(f)$, always 1-formal? This question has led to a certain amount of recent activity, out of which two completely different examples came out, showing that, in general, the Milnor fiber is *not* formal.

Example 10.6 (Zuber [92])**.** Let $\mathcal{A} = \mathcal{A}(3, 3, 3)$ be the monomial arrangement in $\mathbb{C}^3$ defined by the polynomial

$$f(z_0, z_1, z_2) = (z_0^3 - z_1^3)(z_0^3 - z_2^3)(z_1^3 - z_2^3),$$

and let F be the Milnor fiber of f. It is shown in [92] that $TC_1(\mathcal{V}_1(F, C)) \neq \mathcal{R}_1(F, \mathbb{C})$; hence, by Theorem 7.1, F is not 1-formal.

Example 10.7 (Fernández de Bobadilla [46])**.** Consider the polynomial

$$f(z_0, \ldots, z_{10}) = z_0 z_2 z_3 z_5 z_6 + z_0 z_2 z_4 z_7 + z_1 z_2 z_4 z_8 + z_1 z_3 z_5 z_9 + z_1 z_3 z_4 z_{10}.$$

Then f is weighted homogeneous of degree 5, with weights $(1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2)$. As shown in [46], the Milnor fiber F is homotopy equivalent to the complement of the coordinate subspace arrangement $\mathcal{A} = \mathcal{A}_K$ from [26, §9.3]. This arrangement consists of subspaces $H_1, \ldots, H_5$ in $\mathbb{C}^6$, given by $H_i = \{x_i = x_{i+1} = 0\}$.

It is easy to see that F is 2-connected. Computations from [26, §6.2] show that there are classes $\alpha, \beta, \gamma \in H^3(F, \mathbb{Z}) = \mathbb{Z}^5$ such that the triple Massey product $\langle \alpha, \beta, \gamma \rangle \in H^8(F, \mathbb{Z}) = \mathbb{Z}$ is defined, with 0 indeterminacy, and does not vanish. Hence, F is not formal.

In view of these examples, Question 5.5 from [73] needs to be reformulated, as follows.

Question 10.8. For which (weighted) homogeneous polynomials f is the Milnor fiber $F(f)$ a 1-formal (or even formal) space?

For hyperplane arrangements, we can ask an even more precise question.

Question 10.9. Let $\mathcal{A}$ be a central arrangement in $\mathbb{C}^3$, with Milnor fiber F.
   (i) Is the Milnor fiber 1-formal?
   (ii) If not, can this be detected by means of (triple) Massey products in $H^2(F, \mathbb{Q})$?
   (iii) Is the formality property of the Milnor fiber combinatorially determined?

11. Three-dimensional manifolds

11.1. Alexander polynomial and characteristic varieties. Throughout this section, M will be a compact, connected 3-manifold, possibly with boundary ∂M. Let $G = \pi_1(M)$ be the fundamental group of M, and $H = \mathrm{ab}(G)$ its maximal torsion-free abelian quotient. Let $E_1(A_G) \subset \mathbb{Z}H$ be the Alexander ideal, and $\Delta_G = \gcd(E_1(A_G))$ the Alexander polynomial of G.

We start with a lemma from [**29**], based on the work of Eisenbud–Neumann [**35**] and McMullen [**66**].

Lemma 11.1. *If either (1) $\partial M \neq \emptyset$ and $\chi(\partial M) = 0$, or (2) $\partial M = \emptyset$ and M is orientable, then $E_1(A_G)$ is almost principal (in the sense of Definition 4.14).*

In the first case, $E_1(A_G) = I_H \cdot (\Delta_G)$, by [**35**], while in the second case, $I_H^2 \cdot (\Delta_G) \subseteq E_1(A_G)$, by [**66**]. Proposition 4.15 now yields the following corollary, relating the first characteristic variety of M to the zero set of the Alexander polynomial of $G = \pi_1(M)$.

Corollary 11.2. *Under the above assumptions,*

    (i) $\left(\mathcal{V}_1(G) \cap \mathrm{Hom}(G, \mathbb{C}^\times)^0\right) \setminus \{1\} = V(\Delta_G) \setminus \{1\}$.

    (ii) *If, moreover, $H_1(G, \mathbb{Z})$ is torsion-free, then $\mathcal{V}_1(G) \setminus \{1\} = V(\Delta_G) \setminus \{1\}$.*

Example 11.3. Let $L = (L_1, \ldots, L_n)$ be a link in S^3, and let $M = S^3 \setminus \bigcup_{i=1}^n N(L_i)$ be its exterior (i.e., the complement of an open tubular neighborhood of the link). Then M is a compact, connected, orientable 3-manifold, with ∂M consisting of n disjoint tori, so that $\chi(\partial M) = 0$.

Let $G = \pi_1(M)$ be the link group. By Lemma 11.1, the Alexander ideal of G is almost principal. Since $H_1(G, \mathbb{Z}) = \mathbb{Z}^n$, Corollary 11.2(ii) insures that $\mathcal{V}_1(G) = V(\Delta_G)$, at least away from 1.

11.2. Resonance varieties. Assume now M is closed (i.e., $\partial M = \emptyset$) and orientable, and fix an orientation $[M] \in H^3(M, \mathbb{Z}) \cong \mathbb{Z}$. With this choice, the cup product on M and Kronecker pairing determine an alternating 3-form η on $H^1(M, \mathbb{Z})$, given by $\eta(x, y, z) = \langle x \cup y \cup z, [M] \rangle$. In turn, the cup-product map $\mu \colon H^1(M, \mathbb{Z}) \wedge H^1(M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ is determined by the form η, via $\langle x \cup y, \gamma \rangle = \mu(x, y, z)$, where $z = \mathrm{PD}(\gamma)$ is the Poincaré dual of $\gamma \in H_2(M, \mathbb{Z})$.

Fix a basis $\{e_1, \ldots, e_n\}$ for $H^1(M, \mathbb{C})$, and choose $\{e_1^\vee, \ldots, e_n^\vee\}$ as basis for $H^2(M, \mathbb{C})$, where $e_i^\vee$ denotes the Kronecker dual of $\mathrm{PD}(e_i)$. Writing $\mu(e_i, e_j) = \sum_{k=1}^n \mu_{ijk} e_k^\vee$ as in (29), we find that $\eta(e_i, e_j, e_k) = \mu_{ijk}$.

By Proposition 5.2, the resonance variety $\mathcal{R}_1(M)$ is the vanishing locus of the codimension 1 minors of the $n \times n$ matrix Θ with entries $\Theta_{kj} = \sum_{i=1}^n \mu_{ijk} x_i$. Since η is an alternating form, Θ is a skew-symmetric matrix. Using this fact, the following structure theorem is proved in [**32**].

Theorem 11.4 ([**32**]). *Let M be a closed, orientable 3-manifold. Then:*

    (i) $H^1(M, \mathbb{C})$ *is not 1-isotropic.*

    (ii) *If $b_1(M)$ is even, then $\mathcal{R}_1(M) = H^1(M, \mathbb{C})$.*

    (iii) *If $\mathcal{R}_1(M)$ contains a 0-isotropic hyperplane, then $\mathcal{R}_1(M) = H^1(M, \mathbb{C})$.*

11.3. Thurston norm. For each compact, orientable 3-manifold M, Thurston defined in [**88**] a (semi-)norm on $H^1(M, \mathbb{R})$, as follows. For a cohomology class $\phi \in H^1(M, \mathbb{Z})$, set $\|\phi\| = \inf_S\{\chi_-(S)\}$, with the infimum taken over all properly

embedded, compact surfaces S representing the dual of ϕ, and where $\chi_-(S) = \sum_i \max\{-\chi(S_i), 0\}$, after decomposing S as a disjoint union of connected surfaces S_i.

It turns out that the assignment $\phi \mapsto \|\phi\|$ is a seminorm on $H^1(M, \mathbb{Z})$, which can be extended to the whole of $H^1(M, \mathbb{R})$. Even though $\|\cdot\|$ vanishes on spherical and toroidal classes, it is simply called the *Thurston norm* of M. The unit ball in this norm, $B_T = \{\phi \mid \|\phi\| \leq 1\}$, is a rational convex polyhedron, i.e., the intersection of finitely many rationally defined hyperplanes.

A non-zero cohomology class $\phi \in H^1(M, \mathbb{Z})$ is said to be *fibered* if there is a smooth fibration $f \colon M \to S^1$ such that ϕ corresponds to $f_* \colon H_1(M, \mathbb{Z}) \to \mathbb{Z}$ under the isomorphism $H^1(M, \mathbb{Z}) \cong \operatorname{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z})$. In this case, $\|\phi\| = \chi_-(S)$, where $S = f^{-1}(1)$ is the fiber of the fibration. Given a fibered class ϕ, there exists a (top-dimensional) face F of B_T so that ϕ belongs to the open cone $\mathbb{R}_+ \cdot F^\circ$. Such a face F is called a *fibered face*, the reason being that each integral cohomology class in $\mathbb{R}_+ \cdot F^\circ$ is fibered.

11.4. Alexander norm and Thurston norm. In [**66**], McMullen used the Alexander polynomial of $G = \pi_1(M)$ to define a (semi-)norm on $H^1(M, \mathbb{R})$, called the *Alexander norm*. If we write $\Delta_G = \sum n_i g_i$, with $n_i \in \mathbb{Z} \setminus \{0\}$ and $g_i \in \operatorname{ab}(G)$, then the norm of a class $\phi \in H^1(M, \mathbb{R})$ is defined by $\|\phi\|_A = \sup \phi(g_i - g_j)$. The unit ball B_A of the Alexander norm is, up to scaling, the polytope dual to the Newton polytope of Δ_G.

Theorem 11.5 ([**66**]). *Let M be a compact, orientable 3-manifold whose boundary, if any, is a union of tori. Then:*

    (i) *If $b_1(M) \geq 2$, then $\|\phi\|_A \leq \|\phi\|_T$, for all $\phi \in H^1(M, \mathbb{R})$.*
    (ii) *If $b_1(M) = 1$, then $\|\phi\|_A \leq \|\phi\|_T + b_3(M) + 1$, for all primitive classes $\phi \in H^1(M, \mathbb{Z})$.*

Moreover, equality holds when $\phi \in H^1(M, \mathbb{Z})$ is a fibered class, represented by a fibration $f \colon M \to S^1$ with $\chi(f^{-1}(1)) \leq 0$.

McMullen's inequalities generalize the classical relation $\deg \Delta_K(t) \leq 2 \operatorname{genus}(K)$ for knots. In many cases, the Alexander and Thurston norms agree on all of $H^1(M, \mathbb{R})$. Yet even when M fibers over the circle and $b_1(M) \geq 2$, the two norms may disagree, as examples of Dunfield [**33**] show. Nevertheless, as shown by Long [**61**], the two norms coincide for irreducible graph links in homology 3-spheres.

11.5. BNS invariant. In [**8**], Bieri, Neumann, and Strebel found a remarkable connection between the Thurston norm of M and the BNS invariant of $G = \pi_1(M)$. We summarize their result, as follows.

Theorem 11.6 ([**8**]). *Let M be a compact, connected 3-manifold, with fundamental group $G = \pi_1(M)$. Then, $\Sigma^1(G)$ is the union of open cones $\mathbb{R}_+ \cdot F^\circ$, with F running through the fibered faces of the Thurston unit ball B_T. Consequently:*

    (i) $\Sigma^1(G) = -\Sigma^1(G)$.
    (ii) *M fibers over the circle if and only if $\Sigma^1(G) \neq \emptyset$.*

We exploit this theorem to give the example promised in §7.5, showing that the inclusion $\Sigma^1(G) \subseteq \mathcal{R}_1(G, \mathbb{R})^\complement$ may be strict, even when G is 1-formal and $\Sigma^1(G) = -\Sigma^1(G)$.

Example 11.7. Let $L = (L_1, L_2)$ be a non-fibered, two-component link in S^3 with non-zero linking number. Let X be the complement of L, and $G = \pi_1(X)$. Since $\operatorname{lk}(L_1, L_2) \neq 0$, we have a ring isomorphism $H^*(X, \mathbb{Q}) \cong H^*(T^2, \mathbb{Q})$. Consequently, X is a formal space, and $\mathcal{R}_1(G, \mathbb{R}) = \{0\}$. On the other hand, since G is a link group, $\Sigma^1(G) = -\Sigma^1(G)$, by Theorem 11.6(i). Furthermore, since L is not a fibered link, $\Sigma^1(G) \neq \{0\}^\complement$, by Theorem 11.6(ii).

Here is a another application.

Corollary 11.8 ([**74**])**.** *Let M be a closed, orientable 3-manifold. If $b_1(M)$ is even, and $G = \pi_1(M)$ is 1-formal, then M does not fiber over the circle.*

Indeed, by Theorem 11.4(ii), we must have $\mathcal{R}_1(G, \mathbb{R}) = H^1(G, \mathbb{R})$. Hence, by Corollary 7.5, $\Sigma^1(G) = \emptyset$, and the conclusion follows from Theorem 11.6(ii).

11.6. Kähler 3-manifold groups. The following question was raised by S. Donaldson and W. Goldman and in 1989, and independently by A. Reznikov in 1993: Which 3-manifold groups are Kähler groups? In [**77**], Reznikov obtained a deep restriction on certain groups lying at the intersection of these two classes of fundamental groups. In [**32**], the question was answered in full generality.

Theorem 11.9 ([**32**])**.** *Let G be the fundamental group of a closed 3-manifold. Then G is a Kähler group if and only if G is a finite subgroup of $\mathrm{O}(4)$, acting freely on S^3.*

The backward implication follows from classical work of J.-P. Serre. In the case when $b_1(G) > 0$, the forward implication can be readily explained, using the techniques outlined in §8.4 and §11.2. By passing to orientation double covers if necessary, we may assume $G = \pi_1(M)$, with M a closed, orientable 3-manifold. Then G is still a Kähler group, and so $b_1(G)$ must be even (and positive). From Theorem 11.4, we know that $\mathcal{R}_1(G, \mathbb{C}) = H^1(G, \mathbb{C})$, and $H^1(G, \mathbb{C})$ is not 1-isotropic. But, since G is a Kähler group, this contradicts Theorem 8.5(iv).

To deal with the remaining case, $b_1(G) = 0$, one needs some deep facts from 3-manifold theory, together with theorems of Reznikov and Fujiwara, relating the Kähler, respectively the 3-manifold condition on a group to Kazhdan's property (T).

A similar classification of quasi-Kähler, 3-manifold groups seems to be beyond reach at the present time. Nevertheless, such groups are classified in [**31**, Theorem 1.1], but only at the level of Malcev completions, and under a formality assumption.

Theorem 11.10 ([**31**])**.** *Let G be the fundamental group of a closed, orientable 3-manifold. Assume G is 1-formal. Then the following are equivalent.*

   (i) *$\mathfrak{m}(G) \cong \mathfrak{m}(\pi_1(X))$, for some quasi-Kähler manifold X.*
   (ii) *$\mathfrak{m}(G) \cong \mathfrak{m}(\pi_1(M))$, where M is either S^3, $\#^n S^1 \times S^2$, or $S^1 \times \Sigma_g$.*

In particular, if G is a quasi-Kähler, 1-formal, orientable 3-manifold group, then its Malcev completion is isomorphic to that of either 1, F_n, or $\mathbb{Z} \times \pi_1(\Sigma_g)$.

11.7. Boundary manifolds of line arrangements. We conclude with a more detailed description of a special class of closed, orientable 3-manifolds, arising in the context of hyperplane arrangements. Let $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$ be an arrangement of lines in $\mathbb{CP}^2$. The *boundary manifold* $M = M(\mathcal{A})$ is obtained by taking the boundary of a regular neighborhood of $\bigcup_{i=0}^n \ell_i$ in $\mathbb{CP}^2$. Such manifolds were

investigated by Jiang and Yau [53], Westlund [90], and Hironaka [52], and more recently, in [19, 20].

Example 11.11 ([19]). Suppose $\mathcal{A}$ is a pencil of $n+1$ lines; then $M = \sharp^n S^1 \times S^2$. On the other hand, if $\mathcal{A}$ is a near-pencil (i.e., a pencil of n lines, together with another line in general position), then $M = S^1 \times \Sigma_{n-1}$.

Suppose now $\mathcal{A}$ has r non-transverse intersection points, i.e., points $F_J = \bigcap_{j \in J} \ell_j$ of multiplicity $|J| \geq 3$. Blowing up $\mathbb{CP}^2$ at each of these points, we obtain an arrangement $\widetilde{\mathcal{A}} = \{L_0, \ldots, L_{n+r}\}$ in $\widetilde{\mathbb{CP}}^2 \cong \mathbb{CP}^2 \# r\overline{\mathbb{CP}}^2$, consisting of the proper transforms of the lines of $\mathcal{A}$, together with the exceptional lines arising from the blow-ups.

This construction realizes the boundary manifold of $\mathcal{A}$ as a graph manifold, in the sense of Waldhausen. The underlying graph Γ has vertex set $\mathcal{V}(\Gamma)$, with vertices in one-to-one correspondence with the lines of $\widetilde{\mathcal{A}}$: the vertex corresponding to ℓ_i is labeled v_i, while the vertex corresponding to F_J is labeled v_J. The graph Γ has an edge $e_{i,j}$ from v_i to v_j, $i < j$, if the corresponding lines ℓ_i and ℓ_j are transverse, and an edge $e_{J,i}$ from v_J to v_i if $\ell_i \supset F_J$.

Since M is a graph manifold, the group $G = \pi_1(M)$ may be realized as the fundamental group of a graph of groups. As shown in [20, Proposition 3.7], the resulting presentation for G may be simplified to a commutator-relators presentation.

Theorem 11.12 ([20]). *Let $\mathcal{A}$ be an essential line arrangement in $\mathbb{CP}^2$, let Γ be the associated graph, and let G be the fundamental group of the boundary manifold $M(\mathcal{A})$. Then the Alexander polynomial of G is*

$$\Delta_G = \prod_{v \in \mathcal{V}(\Gamma)} (t_v - 1)^{m_v - 2},$$

where m_v denotes the degree of the vertex v, and $t_v = \prod_{i \in v} t_i$.

11.8. Jumping loci of boundary manifolds. The first characteristic variety of the group $G = \pi_1(M(\mathcal{A}))$ can be easily computed from Theorem 11.12. First note that $\Delta_G(1) = 0$. Hence, by Corollary 11.2(ii),

$$(50) \qquad \mathcal{V}_1(G) = \bigcup_{v \in \mathcal{V}(\Gamma) \,:\, m_v \geq 3} \{t_v - 1 = 0\}.$$

The resonance varieties of boundary manifolds were studied in detail in [19, 20]. As before, let $\mathcal{A} = \{\ell_0, \ldots, \ell_n\}$ be an arrangement of lines in $\mathbb{CP}^2$. The first resonance variety of $G = \pi_1(M(\mathcal{A}))$ admits a particularly simple description:

$$(51) \qquad \mathcal{R}_1(G) = \begin{cases} \mathbb{C}^n & \text{if } \mathcal{A} \text{ is a pencil,} \\ \mathbb{C}^{2(n-1)} & \text{if } \mathcal{A} \text{ is a near-pencil,} \\ H^1(G, \mathbb{C}) & \text{otherwise.} \end{cases}$$

The higher-depth resonance varieties, though, can be much more complicated, as the following example extracted from [19, §6.9] illustrates.

Example 11.13. Let $\mathcal{A}$ be an arrangement of 5 lines in $\mathbb{CP}^2$ in general position, and set $G = \pi_1(M(\mathcal{A}))$. Then $H^1(G, \mathbb{C}) = \mathbb{C}^{10}$, and $\mathcal{R}_7(G) = Q \times \{0\}$, where

$$Q = \{z \in \mathbb{C}^6 \mid z_1 z_6 - z_2 z_5 + z_3 z_4 = 0\},$$

which is an irreducible quadric, with an isolated singular point at 0. On the other hand, formula (50) implies that $\mathcal{V}_d(G) \subseteq \{1\}$, for all $d \geq 1$. Consequently, $TC_1(\mathcal{V}_7(G)) \neq \mathcal{R}_7(G)$. The tangent cone theorem now shows that G is not 1-formal.

The above computation indicates that, for arrangements different from those in Example 11.11, the boundary manifold may well be non-formal. As a matter of fact, we have the following result, which summarizes Theorem 9.7 from [20] and Proposition 4.7 from [29], and makes Theorem 11.10 much more precise for this particular class of 3-manifolds.

Theorem 11.14 ([20, 29]). *Let $\mathcal{A} = \{\ell_0, \ldots, \ell_n\}$ be an arrangement of lines in $\mathbb{CP}^2$, and let M be the corresponding boundary manifold. The following are equivalent:*

- (i) *The manifold M is formal.*
- (ii) *The group $G = \pi_1(M)$ is 1-formal.*
- (iii) *The group G is quasi-projective.*
- (iv) *The group G is quasi-Kähler.*
- (v) *$\mathcal{A}$ is either a pencil or a near-pencil.*
- (vi) *M is either $\sharp^n S^1 \times S^2$ or $S^1 \times \Sigma_{n-1}$.*

The crucial implications here are (ii) $\Rightarrow$ (v), which follows from Theorem 7.1, together with formulas (50) and (51), and (iv) $\Rightarrow$ (v), which makes use of Theorem 8.10 and Theorem 11.12.

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